The Quantum 6j Symbol

Junior Topology Siminar

Mustafa Hajij

The quantum group $U_q(sl_2)$

Definition *Let $q \neq 0, 1, -1$ in $\mathbb{C}$. The quantum group $U_q(sl(2))$ is the algebra over $\mathbb{C}$, with the unit element 1, generated by E, F, K and K^{-1}, subject to the relations*

$$
\begin{aligned}
K.K^{-1} &= K^{-1}.K = 1 \\
KE &= qEK, \\
KF &= q^{-1}FK \\
EF - FE &= \frac{K^2 - K^{-2}}{q - q^{-1}}
\end{aligned}
$$

$U_q(sl_2)$-modules

Definition *A* $U_q(sl(2))-$*module is a vector space* V *together with three fixed linear operators* E_V, F_V *and* K_V*, which satisfy*

$$\begin{aligned} K_V.K_V^{-1} &= K_V^{-1}.K_V = 1 \\ K_V E_V &= q E_V K_V \\ K_V F_V &= q^{-1} F_V K_V \\ E_V F_V - F_V E_V &= \frac{K_V^2 - K_V^{-2}}{q - q^{-1}} \end{aligned}$$

$U_q(sl_2)$-modules

Definition *Given two* $U_q(sl(2))-$*modules* V *and* W*,an intertwiner from* V *to* W *is a linear map* $\phi : V \to W$ *such that*

$$\phi \circ E_V = E_W \circ \phi$$

$$\phi \circ F_V = F \circ_W \phi$$

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The set of all morphisms from V *to* W *will be denoted* $Hom_{U_q(sl(2))}(V, W)$.

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For each $n \geq 0$, let V_n the vector space of homogeneous polynomials of degree n in x and y. This space V_n has a basis the monomials $x^n, x^{n-1}y, ..., xy^{n-1}, y^n$ and this basis show that $dim(V_n) = n + 1$.

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In order to verify that the action of E, F, and K defines a $U_q(sl(2))-$module one must check that the $U_q(sl(2))$ are satisfied.

$U_q(sl_2)$-modules

Definition *Let V be a $U_q(sl(2))$−module and λ be a scaler. A vector $v \neq 0$ in V is said to be of weight $\lambda \in \mathbb{C}$ if $Kv = \lambda v$. If we have, in addition, $Ev = 0$ then v is called a highest weight vector.*

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(5) Two finite dimensional weight vector $U_q(sl(2))-$modules generated by highest weight vectors of the same weight are isomorphic.

Some notation..

Let $n \geq 1$ and let $j \in \{n, n-2, ..., -n+2, -n\}$. Let $e_{n,j} := x^{\frac{1}{2}(n+j)} y^{\frac{1}{2}(n-j)}$. In this notation $E\, e_{n,j} = [n-2j] e_{n,j+2}$, $F\, e_{n,j} = [n-2j] e_{n,j-2}$, and $K e_{n,j} = A^j e_{n,j}$

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Theorem *Let $i, j \geq 0$. Then $V_i \otimes V_j = \bigoplus_{|i-j|+1 \leq k \leq i+j-1} V_k$ where the sum runs over all odd $i + j + k$.*

Some morphisms between modules of $U_q(sl_2)$

Let A$\neq 0 \in \mathbb{C}$ be fixed and let $V := V_1 = span\{x, y\}$ be the $U_q(sl(2))-$module defined eariler.

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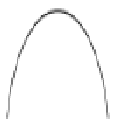
Define  $: V \otimes V \to \mathbb{C}$ via

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Define $\frown : V \otimes V \to \mathbb{C}$ via

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\begin{aligned}
\frown(x \otimes x) &= 0 \\
\frown(x \otimes y) &= iA \\
\frown(y \otimes x) &= -iA^{-1} \\
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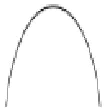
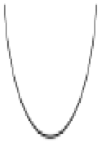
Let A$\neq 0 \in \mathbb{C}$ be fixed and let $V := V_1 = span\{x, y\}$ be the $U_q(sl(2))-$module defined eariler.

Define $\cup : \mathbb{C} \to V \otimes V$ by

$$\cup(1) = iAx \otimes y - iA^{-1} y \otimes x$$

Some morphisms between modules of $U_q(sl_2)$

Lemma *(1) The maps*  *and*  *are intertwiner operators for* $U_q(sl(2))$.

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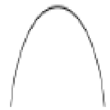
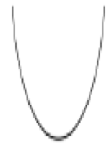
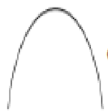
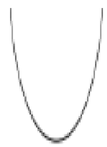
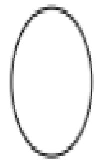
Lemma *(1) The maps*  *and*  *are intertwiner operators for* $U_q(sl(2))$.

(2) The maps  $\circ$  $=$  $: \mathbb{C} \to \mathbb{C}$ *is multiplication by* $-A^2 - A^{-2}$.

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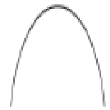
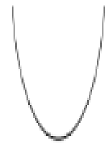
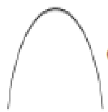
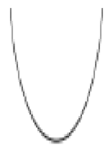
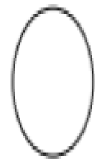
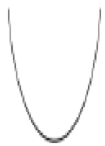
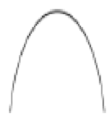
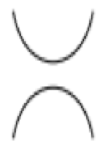
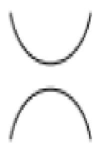
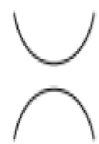
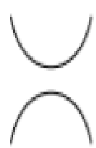
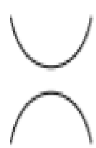
Lemma *(1) The maps*  *and*  *are intertwiner operators for* $U_q(sl(2))$.

(2) The maps  $\circ$  $=$  $: \mathbb{C} \to \mathbb{C}$ *is multiplication by* $-A^2 - A^{-2}$.

(3) The maps  $\circ$  $=$  $: V \otimes V \to V \otimes V$ *is given by*

 $(x \otimes x) \;=\;$  $(y \otimes y) = 0$

 $(x \otimes y) \;=\; -qx \otimes y + y \otimes x$

 $(y \otimes x) \;=\; x \otimes y - q^{-1} y \otimes x$

Some morphisms between modules of $U_q(sl_2)$

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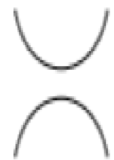
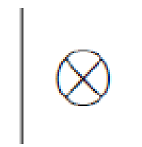
 $:= A[$  $] + A^{-1}[$  $]$

A representation of B_n

Recall the the braid group is given by generators and relations as follows
$B_n =< b_1, ..., b_{n-1} | \ b_i b_j = b_i b_j$ if $|i-j| \geq 2, b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ if $i = 1, 2, ..., n-2 >$

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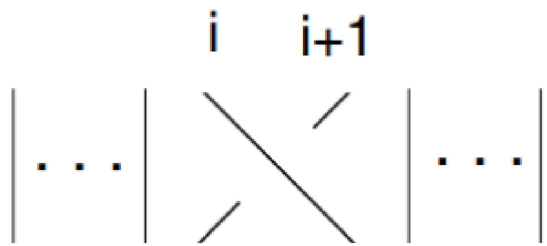
Theorem *There is a representation r_A of B_n on $V^{\otimes n}$ defined by*

$r_A(b_n) =$



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The Temperly-Lieb algebra

The Temperley-Lieb Algebra, denoted by $TL_n(\delta)$ where $\delta = -A^2 - A^{-2}$, is the unital $\mathbb{C}[A, A^{-1}]$ algebra of $U_q(sl(2))$ intertwining maps between n-fold tensor powers of the fundamental representation V. In other words

$$TL_n(\delta) = Hom_{U_q(sl(2))}(V^{\otimes n}, V^{\otimes n})$$

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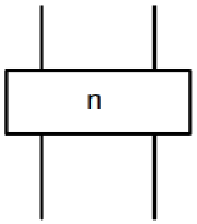
This algebra can be realized as the unital $\mathbb{C}[A, A^{-1}]$ algebra generated by e_i, $1 \leq i \leq n-1$ subject to the relations $e_i e_j = e_j e_i$ if $|i-j| \geq 2$, $e_i e_{i\pm1} e_i = e_i$, and $e_i^2 = -[2]e_i$.

The Jones-Wenztl projectors

Define the Jones-Wenztl projector 

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via the recursive relation:

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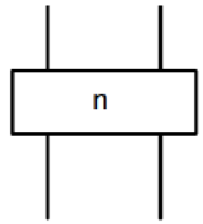
Define the Jones-Wenztl projector 

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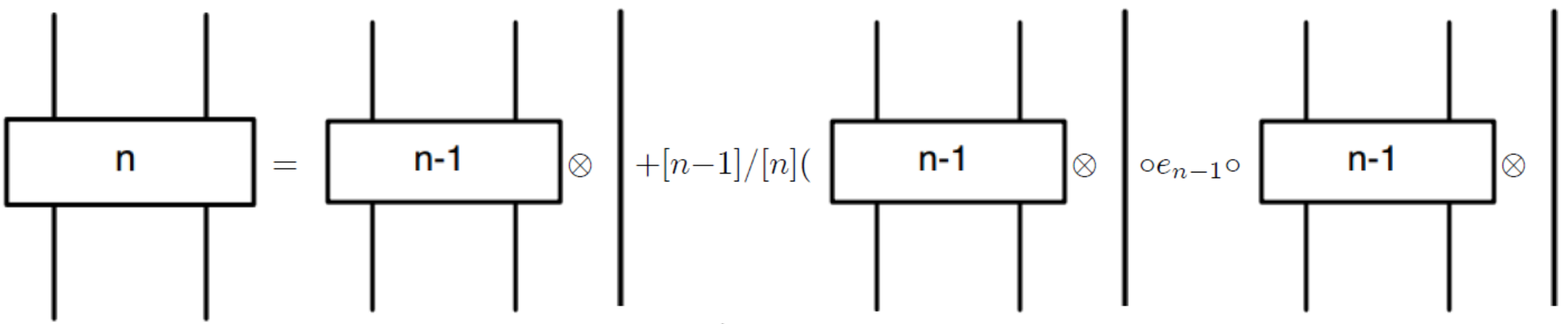
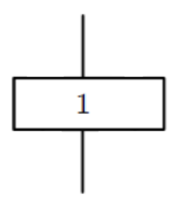
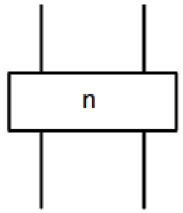
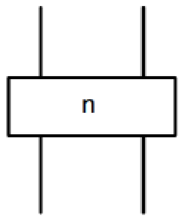


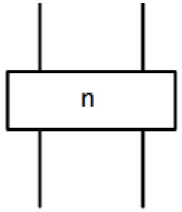
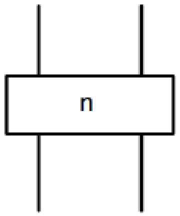


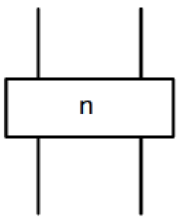
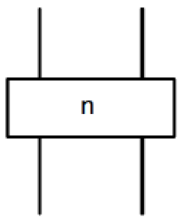
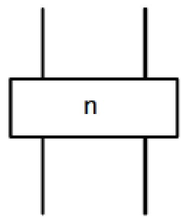
$= id_V$

The Jones-Wenzl projectors

Theorem *Let*  $: V^{\otimes n} \to V^{\otimes n}$ *be the intertwining map defined above, then*  *satisfies the following:*

The Jones-Wenzl projectors

Theorem *Let*  $: V^{\otimes n} \to V^{\otimes n}$ *be the intertwining map defined above, then*  *satisfies the following:*

(1)  $\circ$  $=$ 

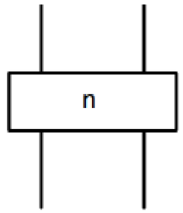
The Jones-Wenztl projectors

Theorem *Let* $\boxed{n} : V^{\otimes n} \to V^{\otimes n}$ *be the intertwining map defined above, then* $\boxed{n}$ *satisfies the following:*

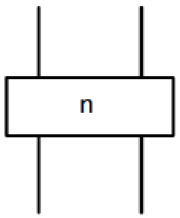
(1) $\boxed{n} \circ \boxed{n} = \boxed{n}$

(2) $\boxed{n} \circ e_i = e_i \circ \boxed{n} = 0$

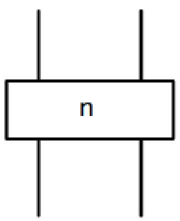
The Jones-Wenztl projectors

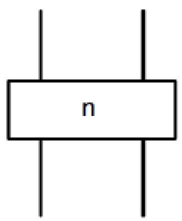
Theorem *Let* 

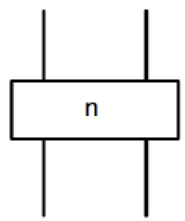
$: V^{\otimes n} \rightarrow V^{\otimes n}$ *be the intertwining map defined*

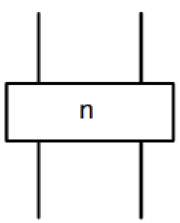
above, then 

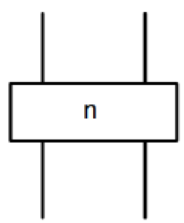
satisfies the following:

(1) 

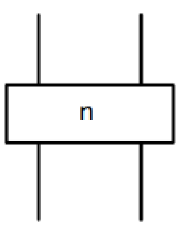
$\circ$ 

$=$ 

(2) 

$\circ\, e_i = e_i \circ$ 

$= 0$

(3) 

is a projector to the subspace $V^{n+1} \subset V^{\otimes n}$.

More morphisms between $U_q(sl_2)$ -modules

Definition *Let* $\bigcup^{1} = \bigcup : \mathbb{C} \to V \otimes V$. *Assume* $\bigcup^{n-1}$ *we define recursivly to be the composition*

More morphisms between $U_q(sl_2)$ -modules

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$$\cup^{n} : \mathbb{C} \to V^{\otimes 2(n-1)} \overset{\rightarrow}{=} V^{\otimes (n-1)} \otimes \mathbb{C} \otimes V^{\otimes (n-1)} \overset{1 \otimes U \otimes 1}{\longrightarrow} V^{\otimes 2n}$$

More morphisms between $U_q(sl_2)$ -modules

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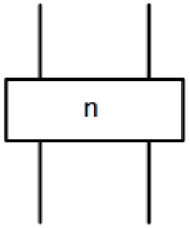
The map $\cap^{n}$ *is defined dually.*

Maps between V_n and the n-fold tensor of the V_2

Definition *Define the map* $\phi_n : V_n \to V^{\otimes n}$ *by*

$$\phi_n(e_{n,j}) = A^{1/4(n+j)(n-j)}$$



$$(x^{\otimes \frac{1}{2}(n+j)} \otimes y^{\otimes \frac{1}{2}(n-j)})$$

here we assume that $[n] \neq 0$.

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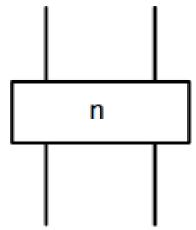


$$(x^{\otimes \frac{1}{2}(n+j)} \otimes y^{\otimes \frac{1}{2}(n-j)})$$

here we assume that $[n] \neq 0$.

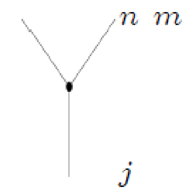
Definition *Let* $n \geq 0$. *Define a map* $\mu_j : V^{\otimes n} \to V_n$ *via*

$$\mu_j(x_1 \otimes ... \otimes x_n) = x_1 \cdot x_2 .... \cdot x_n$$

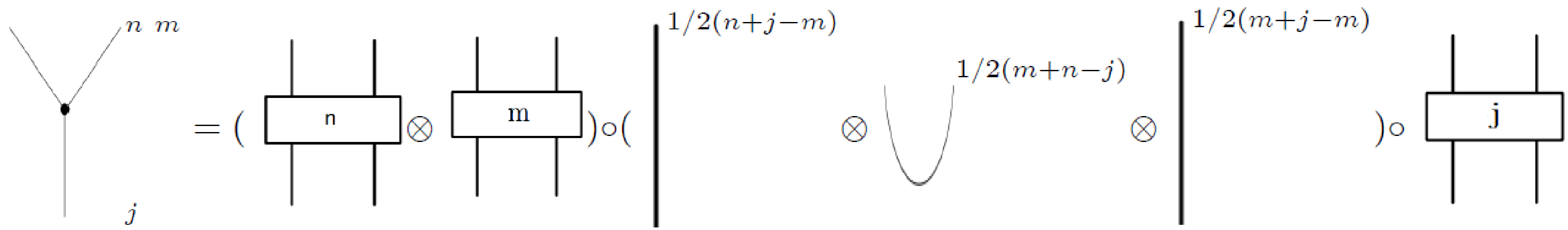
where x_i *are either* x *or* y *and the multiplication on the right occurs in the ring* $\mathbb{C}[x, y]/(xy = qyx)$

Towards defining the 6j Symbol

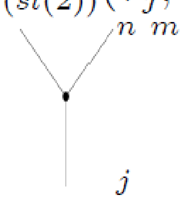
Theorem *Suppose that* (n, m, j) *are admissible triple then the map*



$V^{\otimes j} \to V^{\otimes n} \otimes V^{\otimes m}$ *defined by*



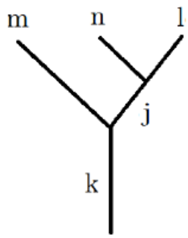
is a $U_q(sl(2))$ *intertwiner. Moreover, The vector space* $Hom_{U_q(sl(2))}(V_j, V_n \otimes V_m)$ *is one dimensional and it is generated by the map* $\mu_n \otimes \mu_m \circ$



$\circ \phi_j$.

Towards defining the 6j Symbol

Theorem *Suppose that* (n, m, j) *are admissible triple then the map*

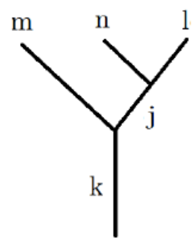


$: V^{\otimes k} \to V^{\otimes m} \otimes V^{\otimes n} \otimes V^{\otimes l}$

defined by

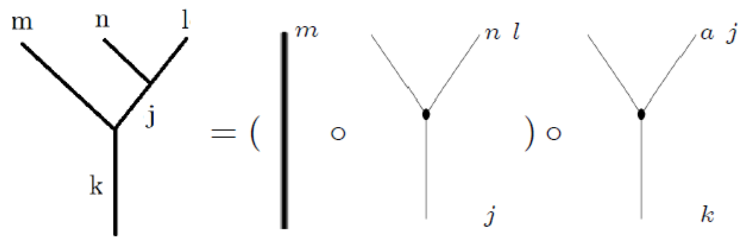
Towards defining the 6j Symbol

Theorem *Suppose that (n, m, j) are admissible triple then the map*



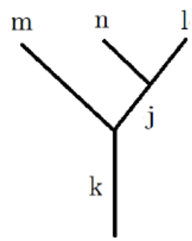
$: V^{\otimes k} \to V^{\otimes m} \otimes V^{\otimes n} \otimes V^{\otimes l}$

defined by



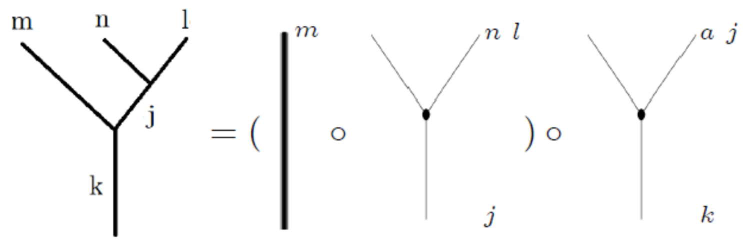
Towards defining the 6j Symbol

Theorem *Suppose that (n, m, j) are admissible triple then the map*



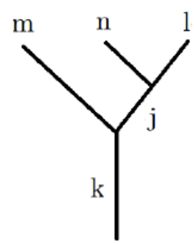
$: V^{\otimes k} \rightarrow V^{\otimes m} \otimes V^{\otimes n} \otimes V^{\otimes l}$

defined by



is a $U_q(sl(2))$ intertwiner. Moreover, the vector space $Hom_{U_q(sl(2))}(V_k, V_m \otimes V_n \otimes V_l)$ is generated by the basis consists of the maps

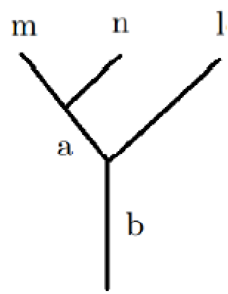
$\{\mu_m \otimes \mu_n \otimes \mu_l \circ$

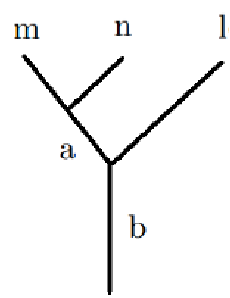


$\circ\ \phi k\}$

where the indices j and k range over all admissible (n, l, j) and (m, j, k).

Towards defining the 6j Symbol

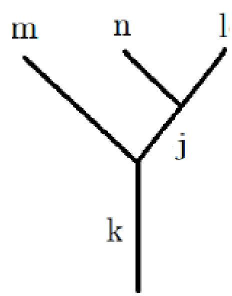
One could show similirly that  has a basis

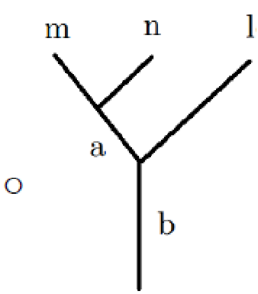
$\{\mu_m \otimes \; \mu_n \otimes \mu_l \circ$  $\circ \; \phi k\}$

with the appropriate admissible colors.

The quantum 6j Symbol

Define the quantum 6j-symbol to be the coefficient $\left\{ \begin{matrix} m & n & a \\ l & k & j \end{matrix} \right\}$ in the following equation

$\mu_m \otimes \mu_n \otimes \mu_l \circ$


$\circ \phi k = \sum_{admissible\ colors} \left\{ \begin{matrix} m & n & a \\ l & k & j \end{matrix} \right\} \mu_m \otimes \mu_n \otimes \mu_l \circ$


$\circ \phi k$

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Thank You