Frobenius algebras and 2D topological quantum field theories

Junior Topology Seminar
Mustafa Hajij

Note: All figures used in this file are from the reference mentioned in the last slide.

Monoidal Categories

A monoidal categories is a category with an additional structure that makes the category behave like a moniod. Namley, we have a "multiplication map" and we have a neutral element.

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We will denote a monoidal category C with a multiplication functor $\otimes$ and a neutral element k by $(C, \otimes, k)$.

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$F(k) = k'$.

Monoidal categories VS strict monoidal categories

Note: what we just called monoidal categories and monoidal functors are actually called in the literature *strict* monoidal categories and *strict* monoidal functors and these latter are actually the ones that one encounters in practice. The difference is that in a strict monoidal category it is not correct to say that $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ but rather $(a \otimes b) \otimes c$ is "naturally isomorphic" to $a \otimes (b \otimes c)$. Something like this should be taken care of whenever you see an equality between objects in a category. All equal signs should be replaced by some isomophisms. We are not going to worry about this here.

Examples of monoidal categories

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$(\boldsymbol{cFA}_{\Bbbk}, \otimes, \Bbbk)$ The category of commutative Frobenius algebras

Oriented cobordism

Let Σ_0 and Σ_1 be closed oriented $(n-1)$-manifolds. An *oriented cobordism* from Σ_0 to Σ_1 is an compact oriented manifold M together with smooth maps

$$\Sigma_0 \to M \leftarrow \Sigma_1$$

such that Σ_0 maps diffeomorphically (preserving orientation) onto the in-boundary of M, and Σ_1 maps diffeomorphically (preserving orientation) onto the out-boundary of M.

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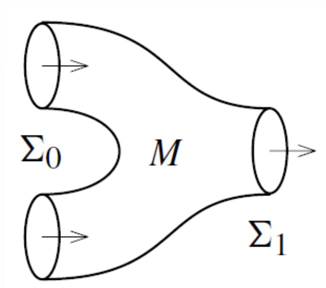
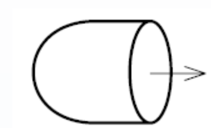




Equivalent cobordisms

Given two oriented cobordisms from Σ_0 to Σ_1,

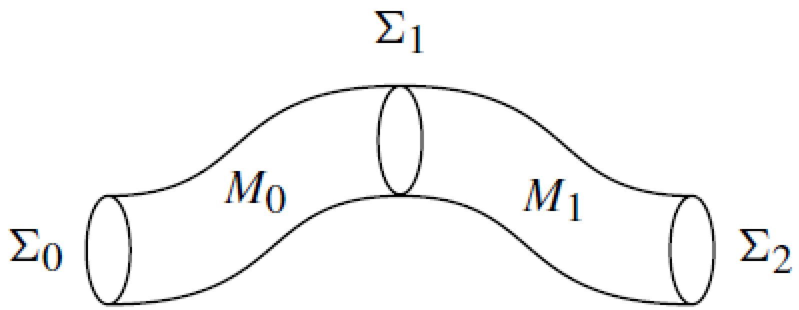
$$\begin{tikzcd} & M' & \\ \Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[ul] \arrow[dl] \\ & M & \end{tikzcd}$$

we say they are *equivalent* if there is an orientation-preserving diffeomorphism $\psi : M \xrightarrow{\sim} M'$ making this diagram commute:

$$\begin{tikzcd} & M' & \\ \Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[ul] \arrow[dl] \\ & M \arrow[uu, "\psi", "\simeq"'] & \end{tikzcd}$$

The composition of two cobordism classes

given specific cobordisms $M_0 : \Sigma_0 \Rightarrow \Sigma_1$ and $M_1 : \Sigma_1 \Rightarrow \Sigma_2$ then there is a well defined diffeomorphism class $M_0 M_1 : \Sigma_0 \Rightarrow \Sigma_2$.



The identity cobordism and invertible cobordisms

The identity cobordism on $\sum$ is the cylinder $\sum \times I$ (or the cobordism class the contains this cobordism)

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$M' : \Sigma_1 \Rightarrow \Sigma_0$ is an inverse to $M : \Sigma_0 \Rightarrow \Sigma_1$ if MM' is the identity on Σ_0 and $M'M$ is the identity on Σ_1.

The Category *nCob*

The objects of $\boldsymbol{nCob}$ are $(n-1)$-dimensional closed oriented manifolds. Given two such objects Σ_0 and Σ_1, then an arrow from Σ_0 to Σ_1 is by definition a diffeomorphism class of oriented cobordisms $M : \Sigma_0 \Rightarrow \Sigma_1$.

Given two cobordisms $M : \Sigma_0 \Rightarrow \Sigma_1$ and $M' : \Sigma_0' \Rightarrow \Sigma_1'$, then likewise we can form their disjoint union $M \coprod M'$ which is naturally a cobordism from $\Sigma_0 \coprod \Sigma_0'$ to $\Sigma_1 \coprod \Sigma_1'$.
We have the empty cobordism $\varnothing_n : \varnothing_{n-1} \Rightarrow \varnothing_{n-1}$.

This is to say, the triple $(\boldsymbol{nCob}, \coprod, \varnothing)$ is a monoidal category

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n- (TQFT) is a rule $\mathscr{A}$ associates

closed oriented $(n-1)$-manifold Σ associates a vector space $\Sigma\mathscr{A}$

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A1: Two equivalent cobordisms must have the same image:

$$M \cong M' \Rightarrow M\mathscr{A} = M'\mathscr{A}.$$

A2: The cylinder $\Sigma \times I$, thought of as a cobordism from Σ to itself, must be sent to the identity map of $\Sigma\mathscr{A}$.

A3: Given a decomposition $M = M'M''$ then

$$M\mathscr{A} = (M'\mathscr{A})(M''\mathscr{A}) \quad \text{(composition of linear maps).}$$

A4: Disjoint union goes to tensor product: if $\Sigma = \Sigma' \coprod \Sigma''$ then $\Sigma\mathscr{A} = \Sigma'\mathscr{A} \otimes \Sigma''\mathscr{A}$. This must also hold for cobordisms: if $M : \Sigma_0 \Rightarrow \Sigma_1$ is the disjoint union of $M' : \Sigma'_0 \Rightarrow \Sigma'_1$ and $M'' : \Sigma''_0 \Rightarrow \Sigma''_1$ then $M\mathscr{A} = M'\mathscr{A} \otimes M''\mathscr{A}$.

A5: The empty manifold $\Sigma = \emptyset$ must be sent to the ground field $\mathbb{k}$. (It follows that the empty cobordism (which is the cylinder over $\Sigma = \emptyset$) is sent to the identity map of $\mathbb{k}$.)

n-TQFT

An *n-dimensional topological quantum field theory* is a monoidal functor from $(\boldsymbol{nCob}, \coprod, \varnothing)$ to $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$

More category theory

A generating set for a category C is a set S of arrows such that every arrow in C can be obtained by composing the arrows of S. A relation is the equality of two ways of writing a given arrow in terms of the generators. A set R of relations is complete if every relation can be obtained by combining the relations in R.

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A generating set for a monoidal category C is a set S of arrows such that every arrow in C can be obtained from the arrows in S by combining composition and "monoidal paralleling".

More category theory

If we have a category C and we take just one object from each isomorphism class in this category then resulting subcategory is called a skeleton of the category C.

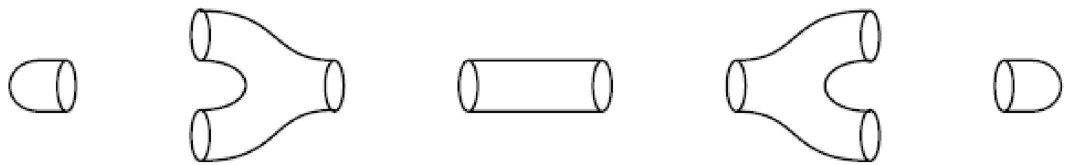
Example : $\{\mathbb{R}, \mathbb{R}^2, ..., \mathbb{R}^n, ...\}$ isskeleton of the category finite dimensional vector spaces over the real numbers.

A Skeleton for *2Cob*

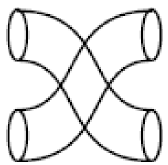
We get a skeleton of $2Cob$ as follows. Let 0 denote the empty 1-manifold; let 1 denote a given circle , and let n denote the disjoint union of n copies of . Then the subcategory $\{0, 1, 2, ...\}$ is a skeleton of $2Cob$. (The arrows are all the possible cobordisms between these objects.)

A generating set for *2Cob*

Proposition. *The monoidal category* **2Cob** *is generated under composition (serial connection) and disjoint union (parallel connection) by the following six cobordisms:*



and



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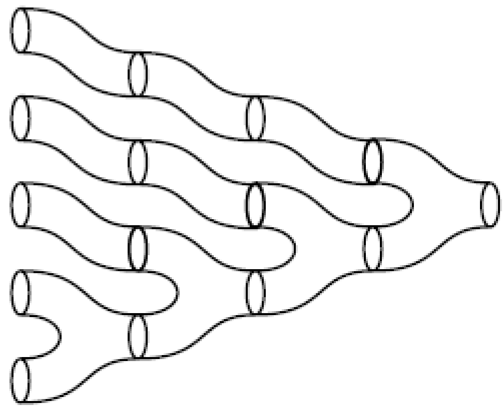
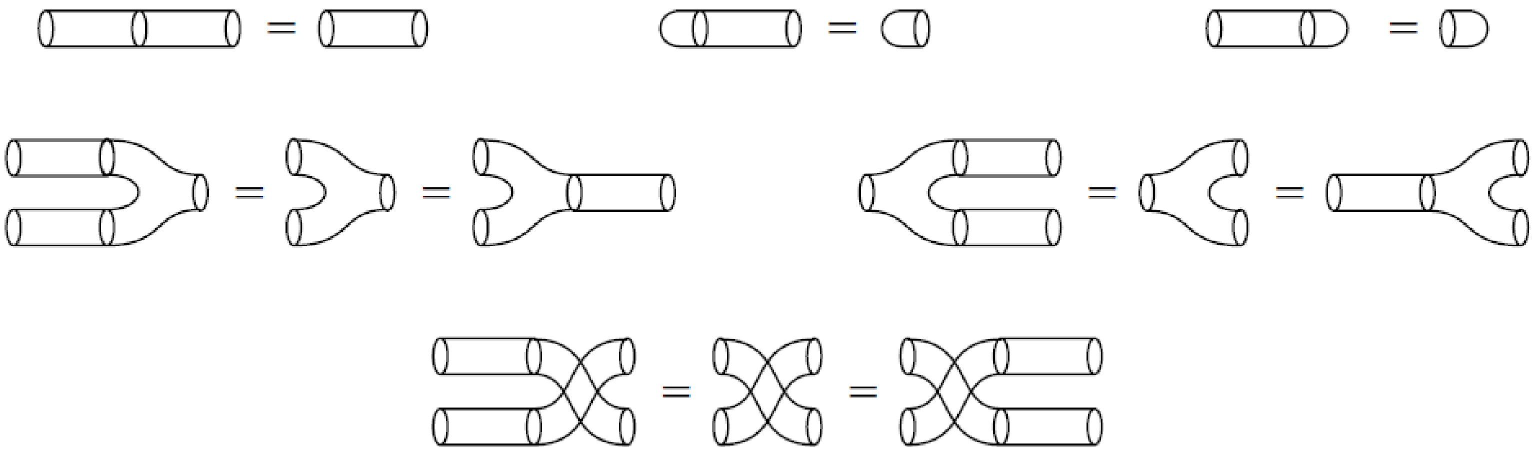


serial connection and parallel connection

Relations for *2Cob*

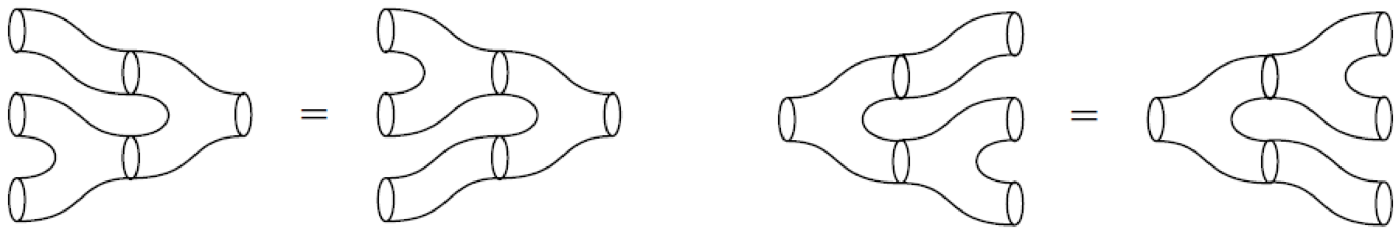
Identity relations



Sewing in discs

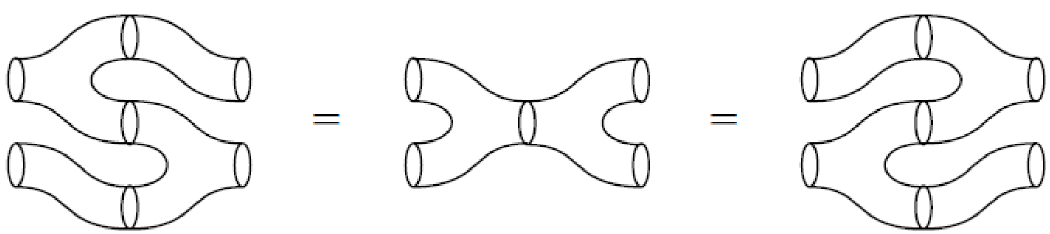
Relations for *2Cob*

Associativity' and 'coassociativity'



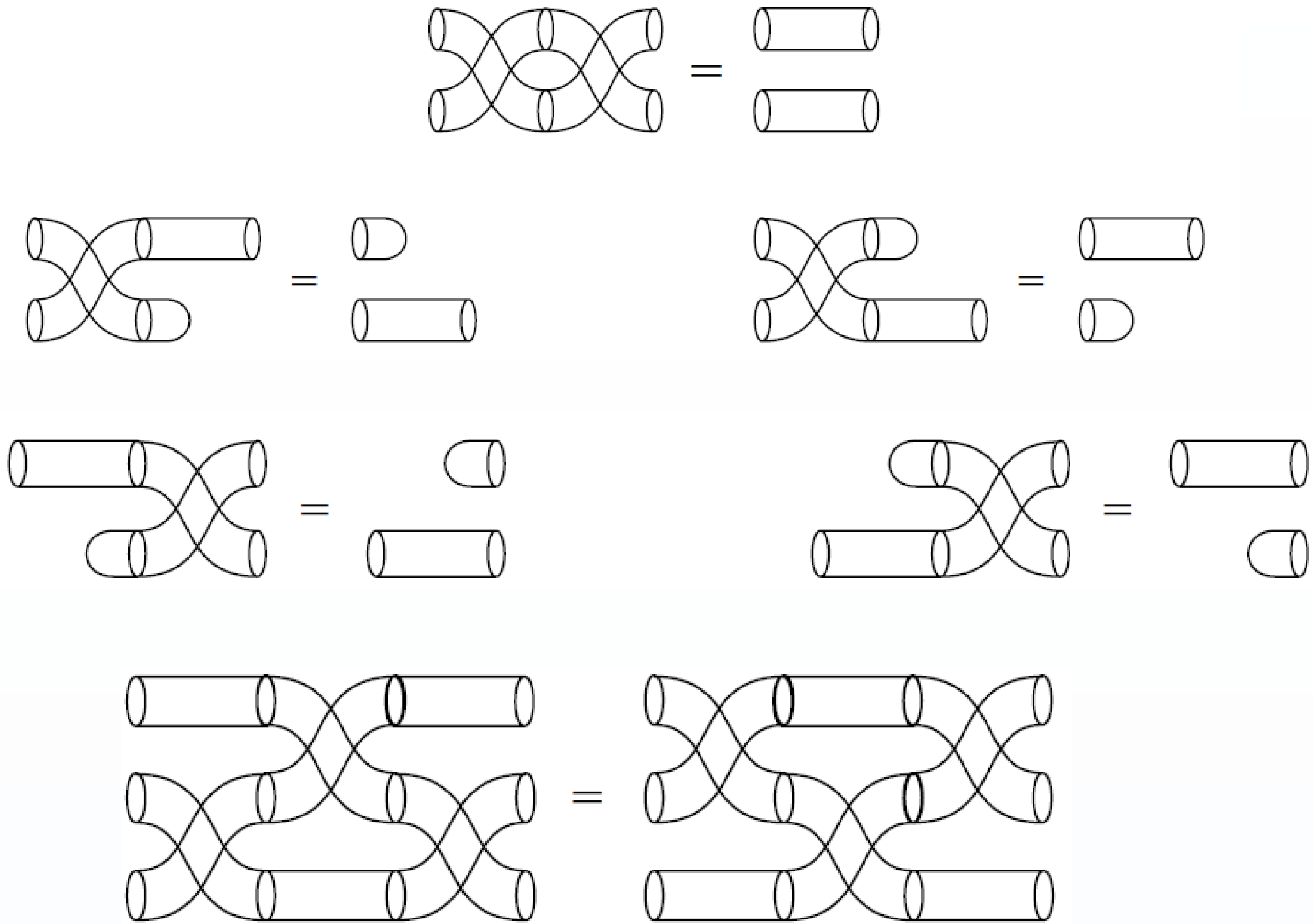
Commutativity' and 'cocommutativity'

'The Frobenius relation'



Relations for *2Cob*

Relations involving the twist

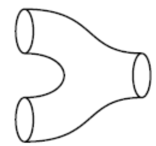


Algebras

A k-algebra is a k-vector space A together with two k-linear maps

η

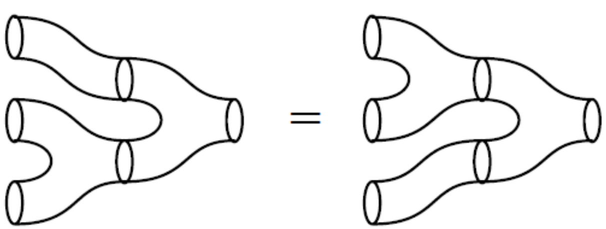
unit



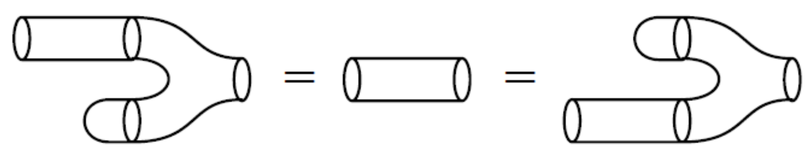
μ

multiplication

such that



associativity



unit axiom

Non-degenerate bilinear map

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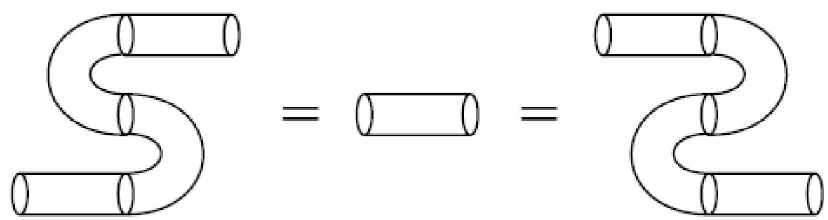
β

Non-degenerate bilinear map

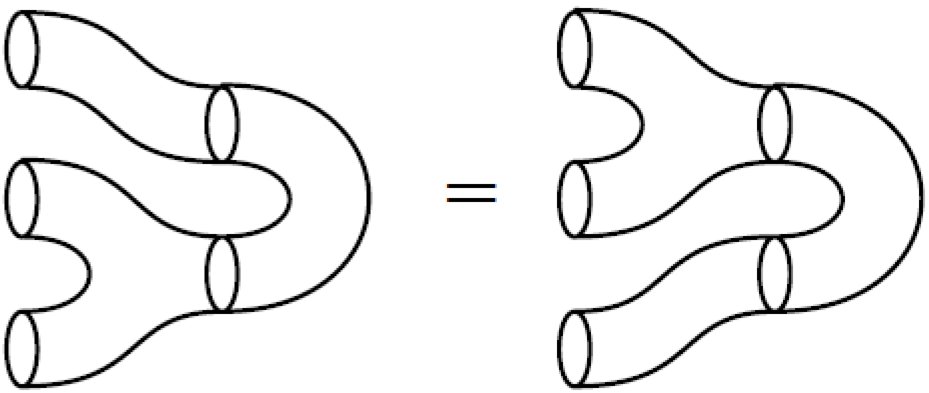
We will denote a bilinear pairing β: $A \otimes A \to k$ by

β

nondegeneracy means *there exists*  *such that* (*copairing* $\gamma : \mathbb{k} \to A \otimes A$)



Associative bilinear map



associativity
of the pairing β

Frobenius algebras

A Frobenius algebra is a k-algebra A of finite dimension, equipped with an associative nondegenerate pairing $\beta\colon A \otimes A \to k$. We call this pairing the Frobenius pairing.

Frobenius algebras

It turns out that the existence of the nondegenerate associative copairing map $k \to A \otimes A$ is equivalent to the existence of a linear map $\epsilon : A \to k$ (whose nullspace contains no nontrivial left ideals)

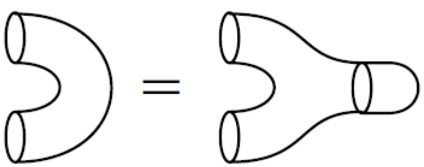
ε

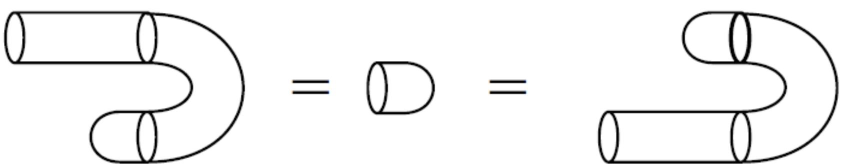
Frobenius form

β

Frobenius pairing

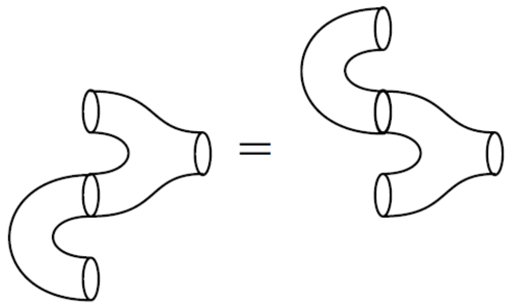
We can draw right away the relation between these two maps:





A Frobenius algebra has a natural coalgebra structure

Lemma. *We have*



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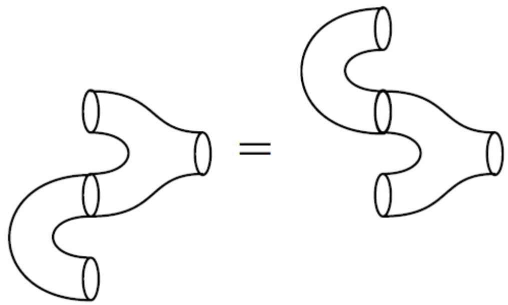
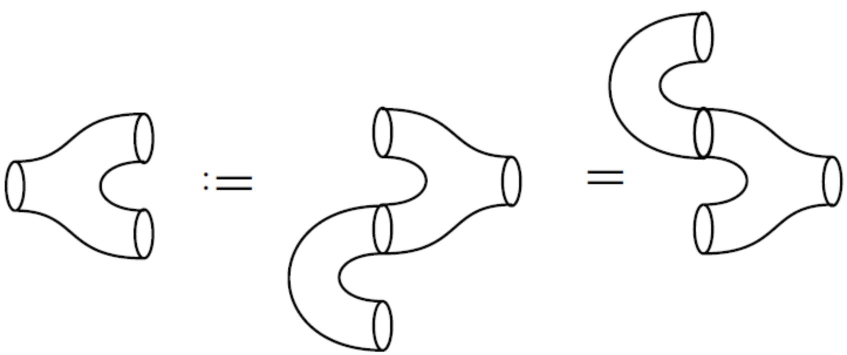


Now define a comultiplication δ



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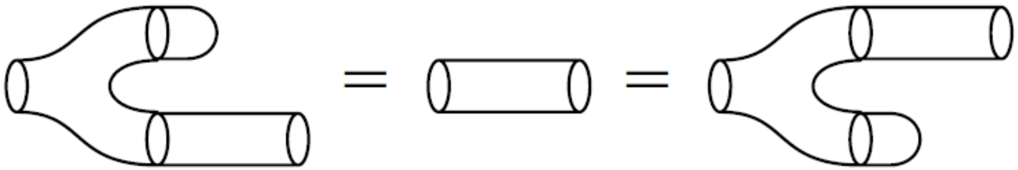
Lemma. *The Frobenius form ε is counit for δ:*



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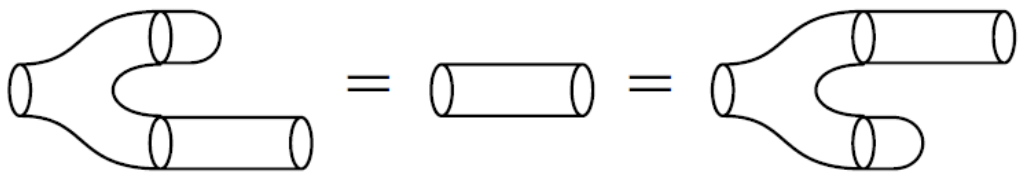
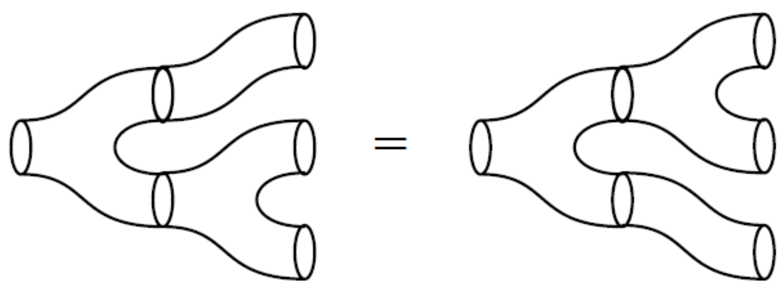
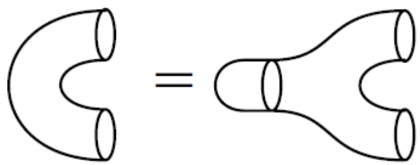


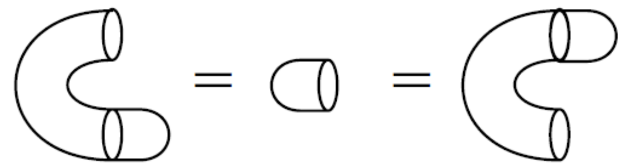
Lemma. *The comultiplication is coassociative:*



More relations between maps

Lemma. *These relations hold:*





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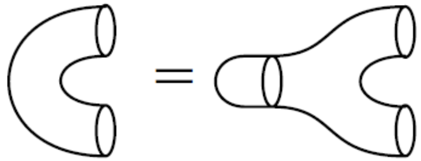
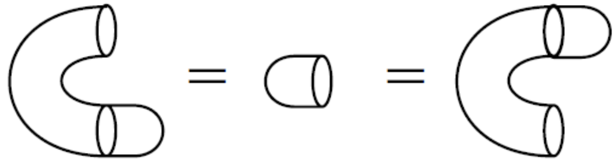
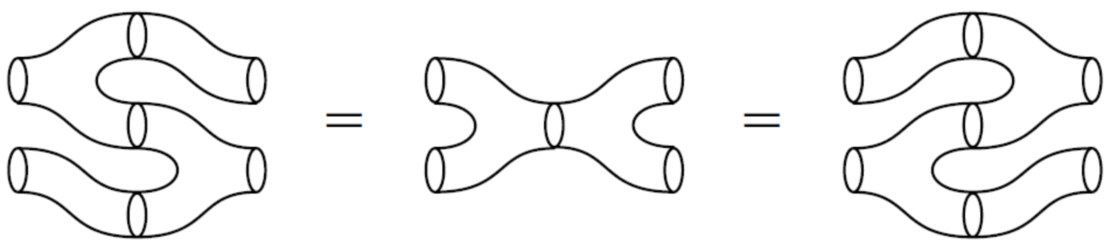




Lemma. *The comultiplication δ defined above satisfies the following relation, called the* Frobenius condition.

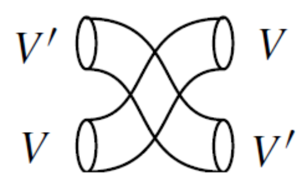


The twist map

The twist map. For every pair of vector spaces V, V' there is a canonical twist map

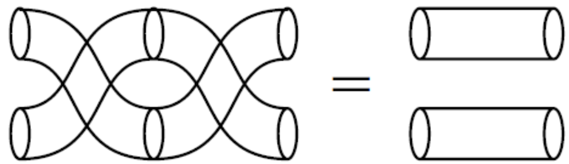
$$\sigma_{V,V'} : V \otimes V' \to V' \otimes V$$

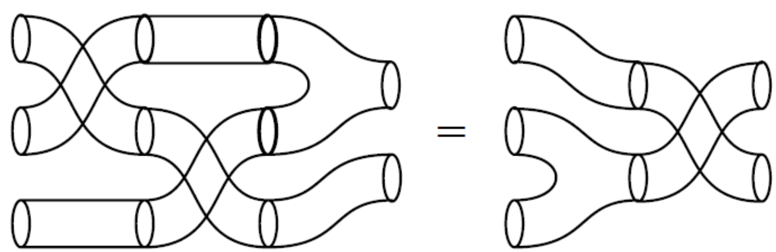
We picture the twist map like



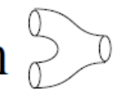
The twist map relations

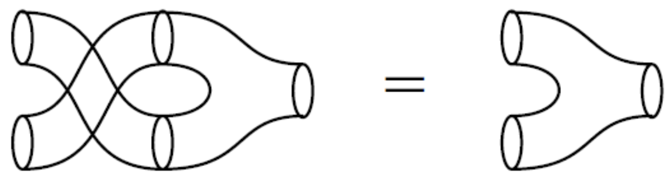
The twist map satisfies



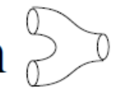


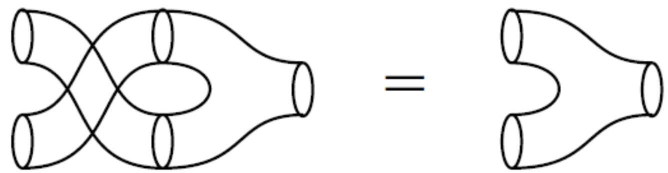
Commutative and co-commutative algebras

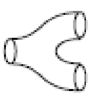
Commutative algebras. Let A be an algebra, with multiplication .
Then we can picture the axiom of being a commutative algebra:

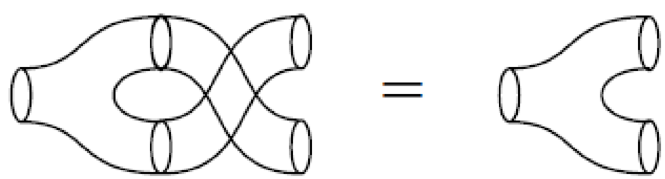


Commutative and co-commutative algebras

Commutative algebras. Let A be an algebra, with multiplication .
Then we can picture the axiom of being a commutative algebra:



Cocommutative coalgebras. A coalgebra A with comultiplication  is said to be *cocommutative* if this relation holds:



The main theorem

A monoidal functor is determined completely by its values on the generators of the source category. In our case we want to specify monoidal functor $A : 2Cob \to Vectk$, so we must specify a vector space A as image of 1, and a linear map for each of the generators.

The main theorem

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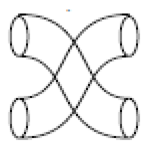
The fact that the functor is monoidal implies in particular that the image of 2 is $A \otimes A$, and so on.

The image of  must be the usual twist for the tensor product,

The main theorem

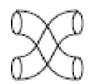
fixe the vector space A:

$$\mathbf{2Cob} \longrightarrow \mathbf{Vect}_{\Bbbk}$$

$$\mathbf{1} \longmapsto A$$

$$\mathbf{n} \longmapsto A^n$$

 $\longmapsto [\mathrm{id}_A : A \to A]$

 $\longmapsto [\sigma : A^2 \to A^2].$

The main theorem

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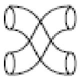
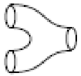
 $\longmapsto [\sigma : A^2 \to A^2].$

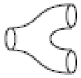
Let the images of the generators be denoted like this:

$$\mathbf{2Cob} \longrightarrow \mathbf{Vect}_{\Bbbk}$$

 $\longmapsto [\eta : \Bbbk \to A]$

 $\longmapsto [\mu : A^2 \to A]$

 $\longmapsto [\varepsilon : A \to \Bbbk]$

 $\longmapsto [\delta : A \to A^2].$

The main theorem

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Conversely, starting with a commutative Frobenius algebra (A, ϵ) (whose multiplication is denoted μ, etc.) then we can construct a TQFT A by using the above description as definition. Also it is clear that these two constructions are inverse to each other. This correspondence also works for arrows (we will not talk about this here) .

The main theorem

Theorem. *There is a canonical equivalence of categories*

$$\boldsymbol{2TQFT}_{\Bbbk} \simeq \mathbf{cFA}_{\Bbbk}.$$

Ref.

J. Kock, *Frobenius algebras and 2D topological quantum field theories*, 2004.

Thank you