Hecke Algebra Representation of Braid Groups and Links Polynomial

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Geometric Topology

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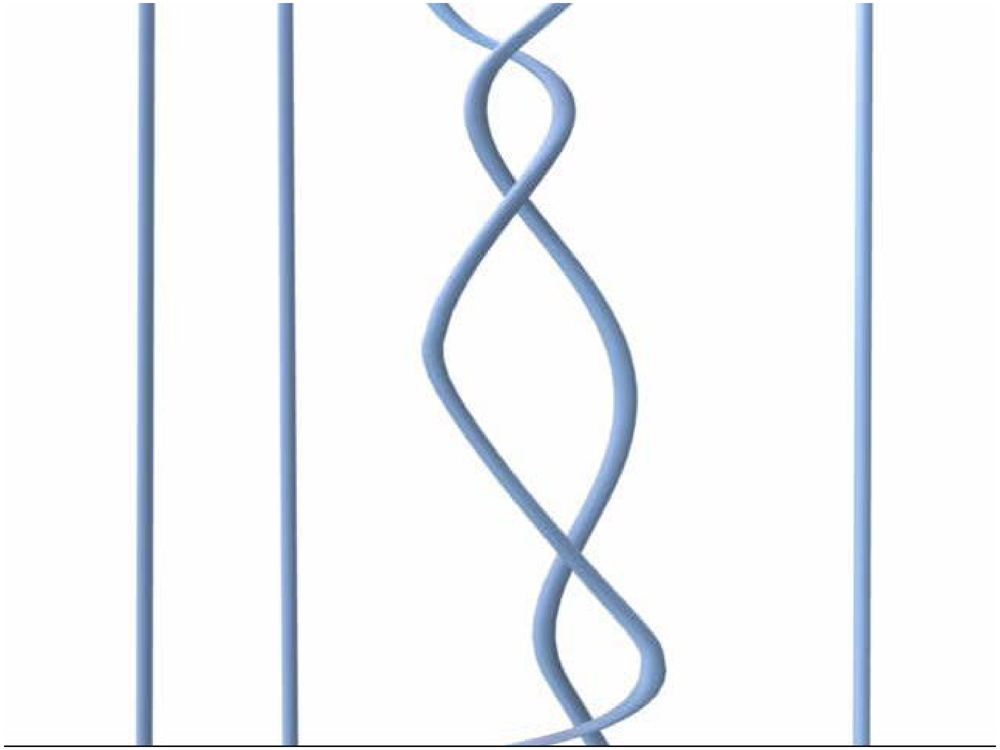
Braids

A Braid Diagram

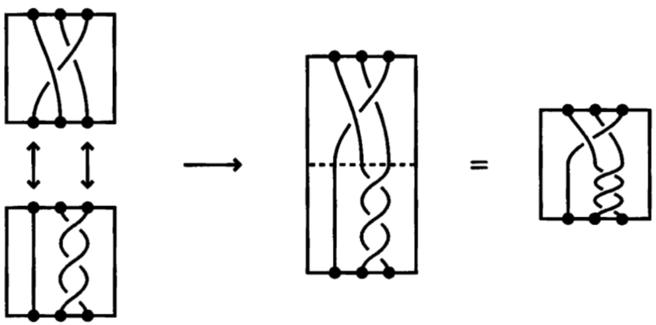
The Braid Group

Let B_n be the set of all equivalence classes of n-braids.

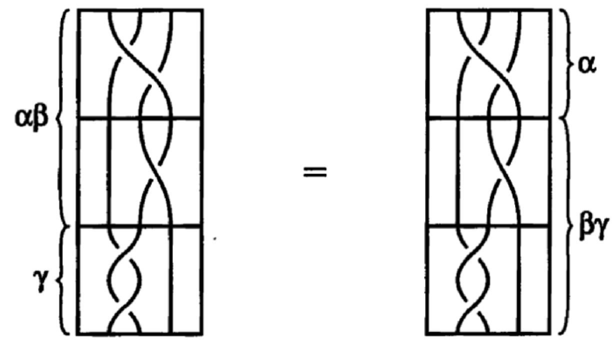


The Braid Group

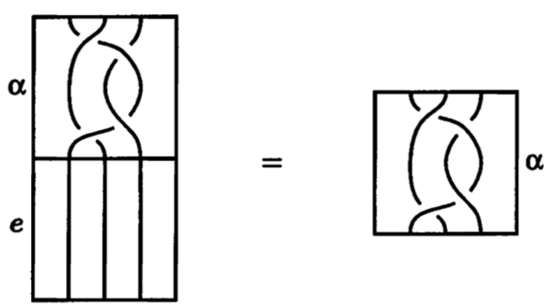
It is possible to define a product in the obvious way on this set.



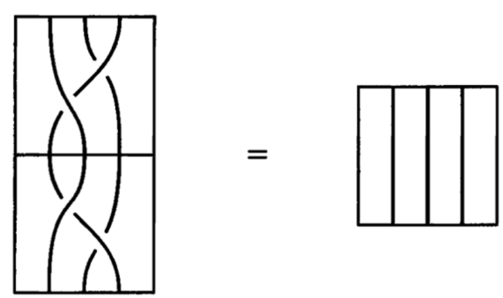
$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$



$$e\alpha = \alpha$$

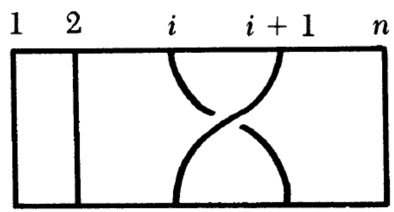


$$\alpha^*\alpha \ = \ e.$$

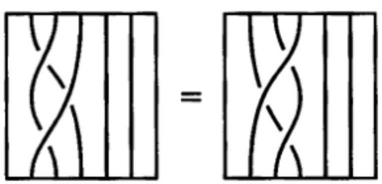


Generators and Relations for B_n

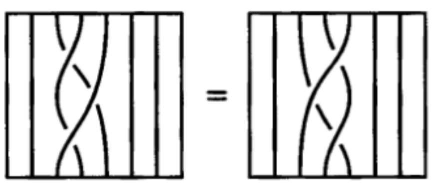
Generators :

$\sigma_i =$ 

Relations :



$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$



$\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$

$\sigma_1\sigma_3 = \sigma_3\sigma_1$

A presentation for The Braid Group

$$B_n = \langle \sigma_1, ..., \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, i = 1, 2, ..., n-2, \sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2 \rangle$$

Examples

$$B_2 = (\sigma_1 \mid \text{———}),$$
$$B_3 = (\sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2),$$
$$B_4 = (\sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3)$$

A presentation for The Braid Group

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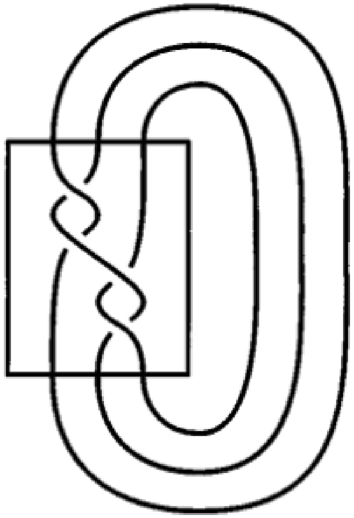
Recall the following simple algebra theorem

Suppose that $G =< S_1 | t = 1$,all $t \in T >$, and if G_2 is a group of matrices. Suppose that $S_2 \subset G_2$, $< S_2 >= G$, and $s \mapsto s'$ isa function from S_1 onto S_2. Suppose further that the generators $s' \in S_2$ satisfy all the relations $t = 1, t \in T$, in the sense that if each $s \in S_1$ is replaced by the corresponding $s' \in S_2$ in each word t$\in T$, then the result is an element $t' \in G_2$ with $t' \in G_2$ with $t' = 1$. Then there is a homomorphism from G_1 onto G_2.

The closure of a braid



Theorem (Alexander) : Any tame oriented link is isotopic to the closure of some braid.

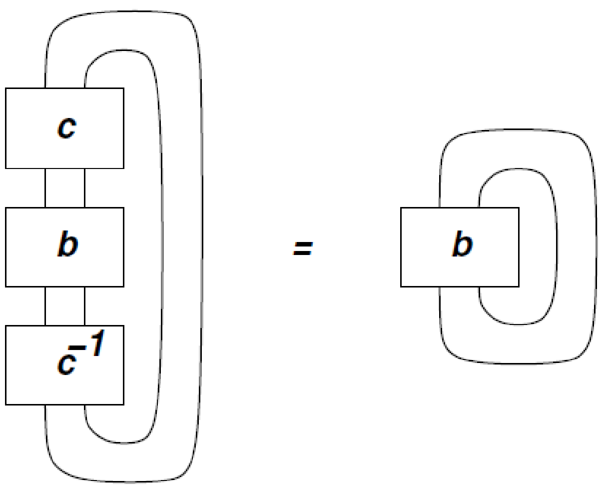
The closure of a braid

A Markov move of type I is changing $\alpha \in B_n$ to $\beta\alpha\beta^{-1} \in B_n$ for any $\beta \in B_n$.

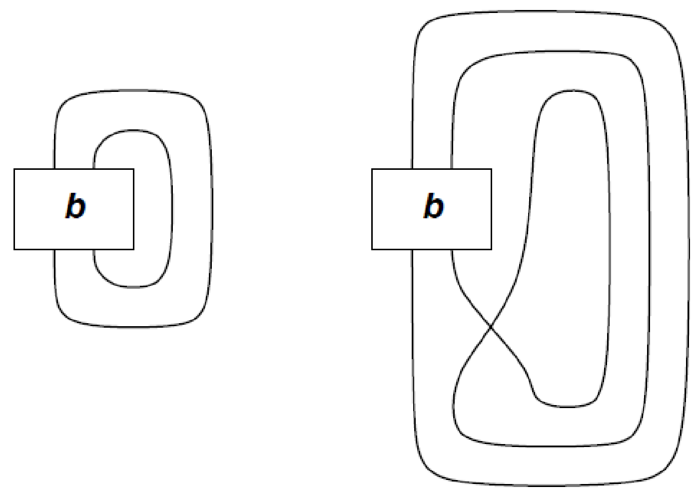
A Markov move of type II is changing $\alpha \in B_n$ to $\beta\sigma_n^{\pm 1} \in B_{n+1}$, or the inverse of this operation.

The representation of a link L as a closed braid is highly non-unique.
If $\alpha \in B_n$ and $\beta \in B_m$ have isotopic closures then there is a finite sequence of Markov moves if type I and II which takes α to β.

Marcov Moves I and II



Marcov Move I



Marcov Move II

Some Observations

The representation of a link L as a closed braid is highly non-unique.
If $\alpha \in B_n$ and $\beta \in B_m$ have isotopic closures then there is a finite sequence of Markov moves if type I and II which takes α to β.

In general if we have a function $\pi : B_n \to X$, where X is some set and we want to check that this function induces an invariant for knots then it is enough to check that this fucntion is invariant under Marcov Moves I and II.

Note that Marcov II is satisfied by two well known functions, namely the trace and the determinant of a matrix so it is natural to think of mapping B_n to some set of matrices to get a knot invariant.

The Burau Representation

- Let t be any non zero complex number. Consider the (n-1) square matrices

$$b_1 = \begin{pmatrix} -t & 0 & & & 0 \\ -1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & 0 & & & 1 \end{pmatrix}, \quad b_i = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & -t & 0 & & \\ & & 0 & -t & 0 & & \\ & & 0 & -1 & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}$$

$$b_{n-1} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & -t \\ 0 & & & 0 & -t \end{pmatrix}$$

$2 \leq i \leq n-2$ where the diagonal $-t$ is in the $i-i$ position.

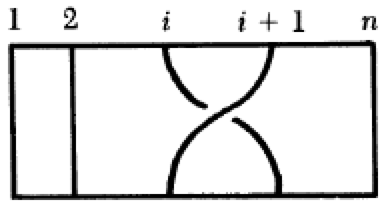
Then it is east to see that there matrices satisfy the relations:

$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ and $b_i b_{i+1} = b_{i+1} b_i$ $|i-j| \geq 2$

So…

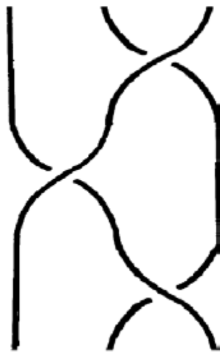
Define the representation $\pi : B_n \longmapsto GL_{n-1}(\mathbb{C})$

$\sigma_i =$



$\longmapsto b_i$

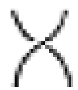
For example $\sigma_2\sigma_1\sigma_2^{-1}$ in B_3 acts on $\mathbb{C}^2$ by:



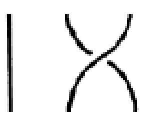
$$= \sigma_2\sigma_1\sigma_2^{-1} \longmapsto b_2 b_1 b_2^{-1}$$

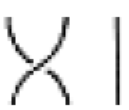
Example of some representations

B_2

 $\rightarrow [-t]$

B_3

 $\rightarrow \begin{bmatrix} 1 & -t \\ 0 & -t \end{bmatrix}$

 $\rightarrow \begin{bmatrix} 1 & -t \\ 0 & -t \end{bmatrix}$

Representation of the symmetric group Sn

Let $G = S_n$, and let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$.

Let S_n act on $\mathbb{C}^n$ in the natural way: if

$$v = \alpha_1 e_1 + \cdots + \alpha_n e_n,$$

then

$$v \cdot \pi = \alpha_1 e_{\pi(1)} + \cdots + \alpha_n e_{\pi(n)}.$$

For instance, if $n = 4$, then under this representation,

$$(143) \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The Way no.23847294879 to Compute The Alexander Polynomial

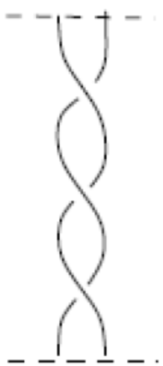
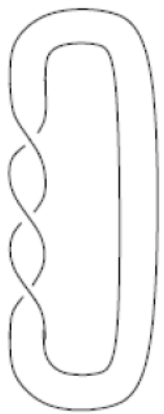
If $\alpha \in B_n$

$$\frac{det(I - \pi(\alpha))}{1 + t + ... + t^{n-1}} = \Delta(\hat{\alpha})$$

Where $\hat{\alpha}$ is the closure of the α.

Note that $det(I - \pi(\alpha))$ is defined using the determinant which allows us to prove the invariance of this expression under Marcov Move I.

Example…The Trefoil

$det(I - \pi($  $)) = det(I - \pi(\sigma_1^3)) = det(1 - (-t^3)) = 1 + t^3 = (1+t)(t^2 - t + 1) = (1+t)\Delta($  $)$

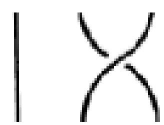
Another Example

Let L be a disjoin union of n circles, then using the fundamental group of the complement of the knot we get

$$\Delta(L) = 0$$

On the other hand we see using Burau representation

$$\det(I - \pi(\sigma_2)) = \det\left(I - \pi\left(\right.\right.$$



$$\left.\left.\right)\right) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -t \\ 0 & -t \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} 0 & -t \\ 0 & 1-t \end{bmatrix} = 0$$

Algebra Given by Generators and Relations

Suppose that A is an algebra over a field K generated by a set of elements $\{x_1, ..., x_q\}$. This means that every element of A can be written as a linear combination of products of the x_i:

$$\sum c_k x_1^{e_{k1}} .....x_q^{ekq}$$

where $c_k \in K, e_{ki} \geq 0.$

There may be many relations that the x_i satisfy which one must know in order to use the generators for computation in A.

Some Examples

Example *Consider the complex numbers $\mathbb{C}$ as an algebra over $\mathbb{R}$. Since $i^2 = -1$, we know that $\mathbb{C}$ is generated by the single element i, and indeed this relation suffices to describe $\mathbb{C}$ in terms of the generator i:*

$$\mathbb{C} \simeq \mathbb{R}[i]/(i^2 + 1).$$

Quotienting out by the ideal generated by i^2+1 is equivalent to the relation $i^2 = -1$ holding in the quotient algebra.

Example *The (real) Quaternions is the four-dimensional vector space over R is generated by the formal elements $\{1, i, j, k\}$, so any element in H looks like $a1 + bi + cj + dk$ and this algebra is subject to the relations*

$$i^2 = j^2 = k^2 = -1$$

$$ijk = -1$$

which means that when we multiply two elements we need ro respect those relations and apply replace them when they show up.

Hecke Algebra H(q,n)

generators

$g_1, g_2, \ldots, g_{n-1}$, and

relations

$$g_i^2 = (q-1)g_i + q,$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},$$

$$g_i g_j = g_j g_i \qquad \text{if } |i-j| \geq 2.$$

Hecke Algebra Representation

We can extend the representation π, $\pi(\sigma_i) = g_i$, to $H(q,n)$ in a natural way. So we have a representaion of The Hecke algebra $H(q,n)$.

The Key Theorem the

THEOREM (Ocneanu). *For every* $z \in \mathbf{C}$ *there is a linear trace* tr *on* $\bigcup_{n=1}^{\infty} H(q, n)$ *uniquely defined by*

1) $\operatorname{tr}(ab) = \operatorname{tr}(ba)$.
2) $\operatorname{tr}(1) = 1$.
3) $\operatorname{tr}(xg) = z \operatorname{tr}(x)$ for $x \in H(q, n)$.

The key observation here is the similarity between the condition 3 in the key theorem and the Markov move II. Property 3 will be used to prove the that the trace is invariant under Markov move 2.

Two Variable Invariant

Definition The two-variable invariant $X_L(q, \lambda)$ of the oriented link L is the function

$$X_L(q, \lambda) = \left(- \frac{1 - \lambda q}{\sqrt{\lambda}(1-q)} \right)^{n-1} (\sqrt{\lambda})^e \mathrm{tr}(\pi(\alpha))$$

where $\alpha \in B_n$ is any braid with $\hat{\alpha} = L$, e being the exponent sum of α as a word on the σ_i's and π the representation of B_n in $H(q, n)$, $\sigma_i \mapsto g_i$.

Example..again the Trefoil

Example The (right-handed) trefoil is given by the closure of the braid $\sigma_1^3 \in B_2$. Thus

$$X_L(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)(\sqrt{\lambda})^3 \mathrm{tr}(g_1^3).$$

By the defining relations of the Hecke algebra we have

$$g_1^3 = (q^2 - q + 1)g_1 + q(q-1) \text{ so that}$$

$$\begin{aligned}
X_L(q,\lambda) &= \left(\frac{\lambda(1-\lambda q)}{1-q}\right)\left((q^2-q+1)\frac{(1-q)}{1-\lambda q}q(q-1)\right) \\
&= \lambda(1+q^2-\lambda q^2) \\
&= (\lambda q)\left(\left(\sqrt{q} - 1/\sqrt{q}\right)^2 + 2 - \lambda q\right) \\
&= (2t^2 - t^4) + t^2x^2.
\end{aligned}$$

Some Properties

- This invariant respects connected sum.
- It is sensitive to the orientation of the link.
- Can distinguish between the knot and its Mirror image.
- The Alexander polynomial is the a special case from this polynomial by setting

$$\Delta(L) = X_L(t, 1/t)$$

- The Hecke algebra representation of B_n can be used to get representations of subgroups of the mapping class groups which can be extended in some cases to the whole group.

Thanks