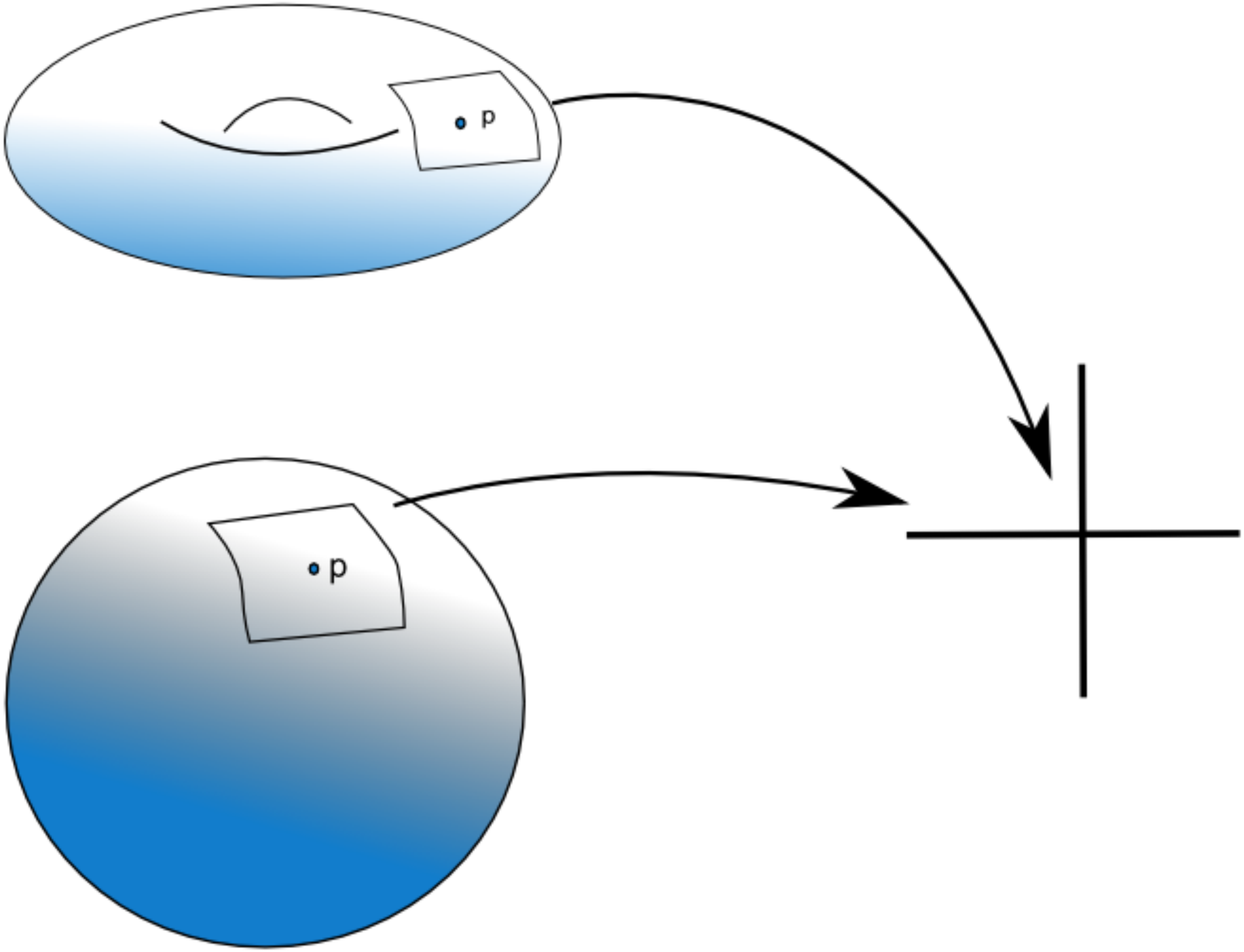
Knots and 3-Manifolds

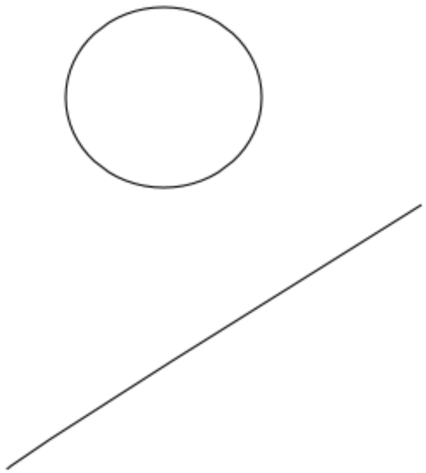
Mustafa Hajij

What is a Manifold?



What is a Manifold?

1-manifolds



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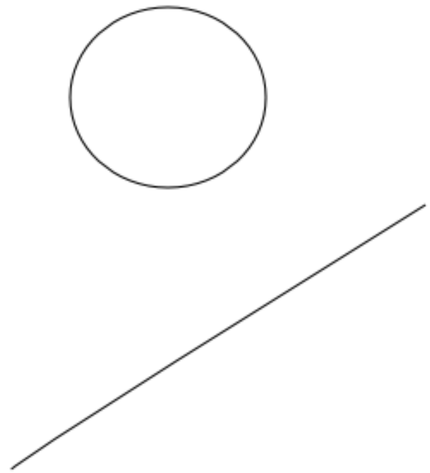
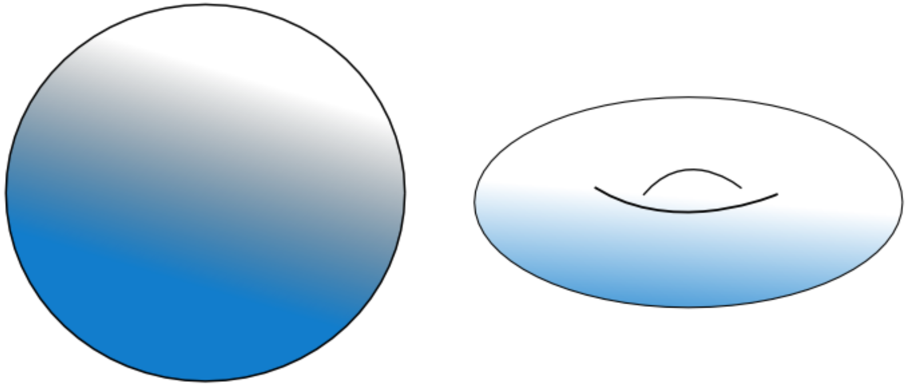


2-manifolds



Classification of Manifolds

We will only consider closed, orientable, and connected manifolds.

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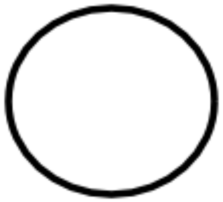

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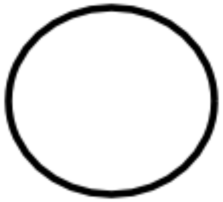
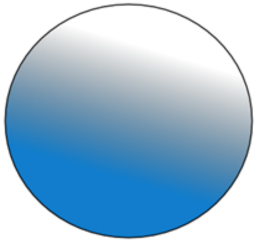


2-manifolds



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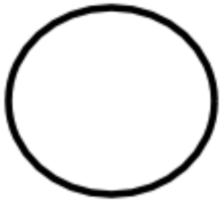

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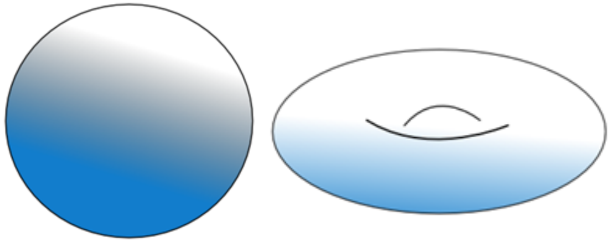

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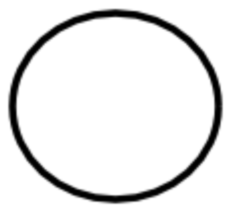

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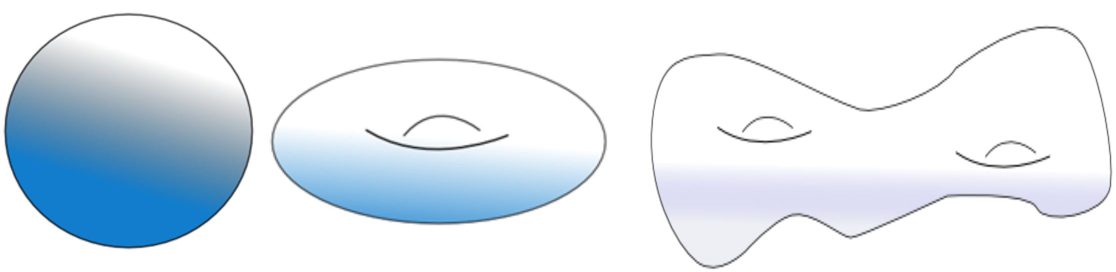

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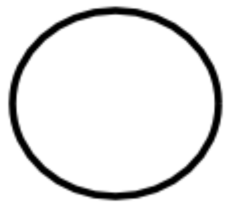

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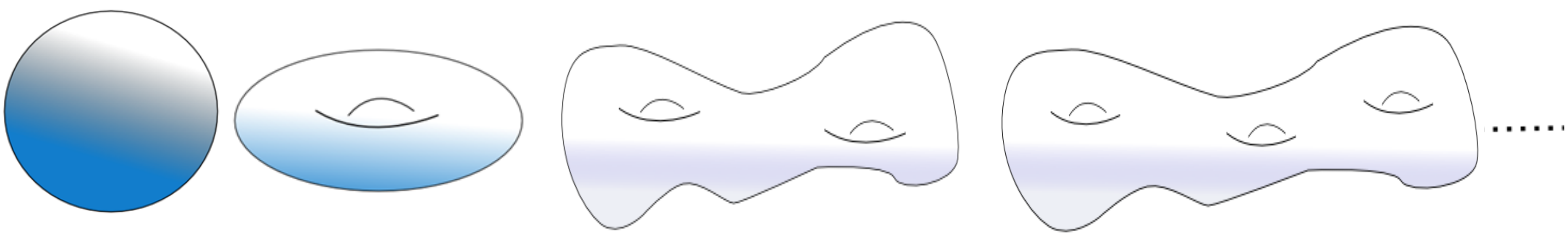

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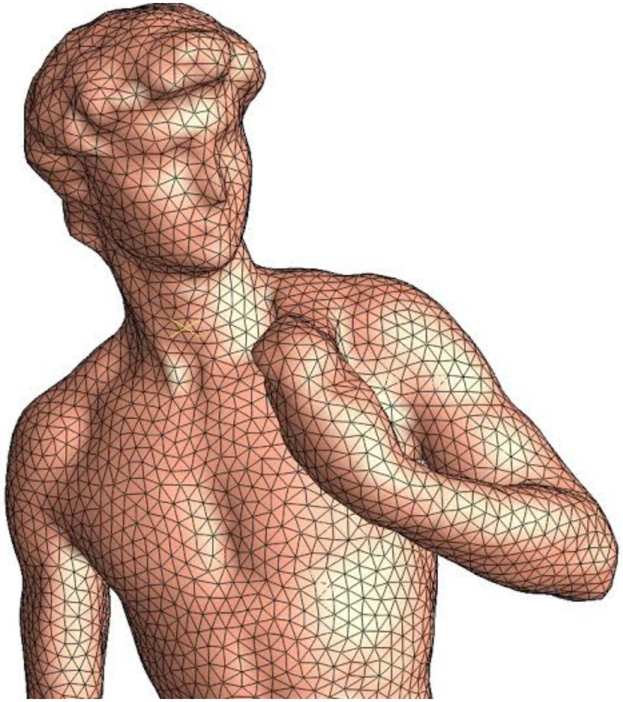
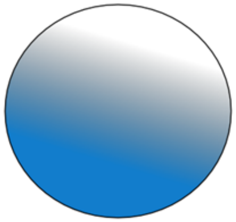


An example of such a quantity is the Euler characteristic.

If we triangulate a surface then this number is given by : **# vertices - # edges + # faces**

Classification of 2-Manifolds

surface	Euler characteristic
	2

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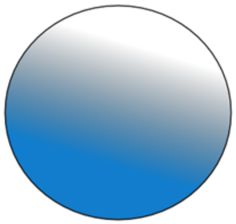
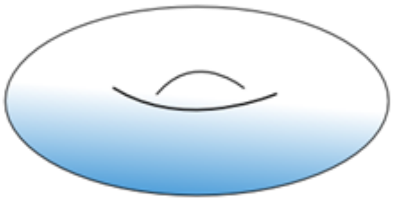
	2
	0

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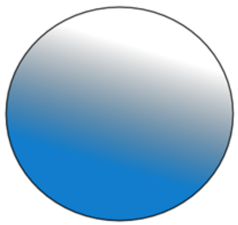
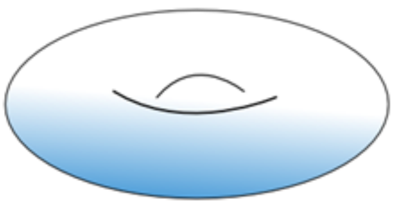
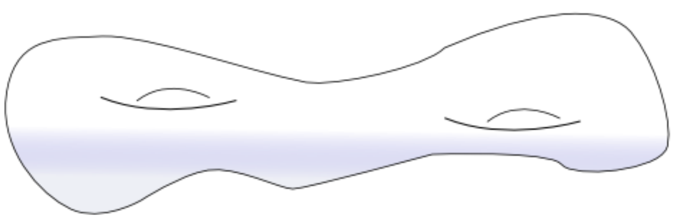
	**2**
	**0**
	**- 2**

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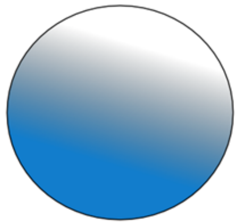
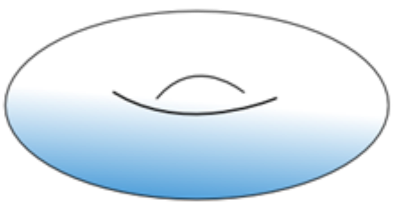
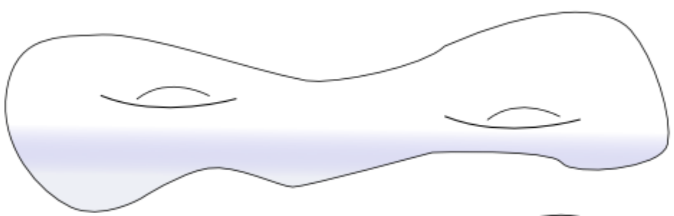
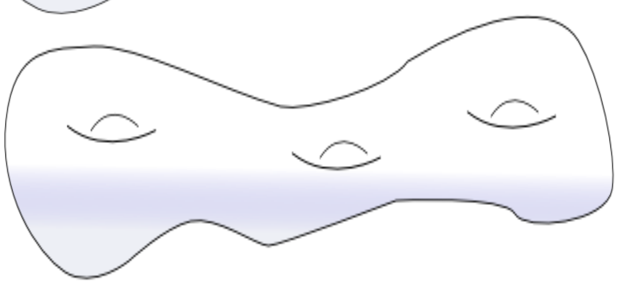
	2
	0
	- 2
	- 4
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However, we have a multiple methods to generate the entire “list” of them.

This list has redundancies and one usually tries to find an effective invariant to distinguish between two given manifolds.

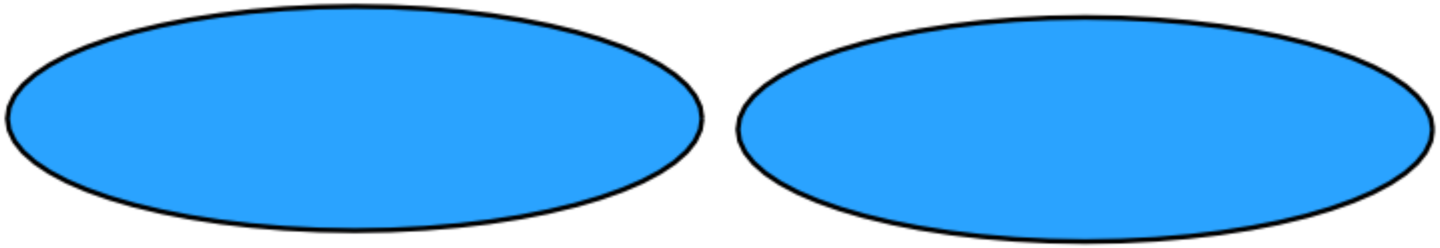
Knots can be used to generate every list of 3-manifolds (closed, oriented)

Examples of 3-manifolds

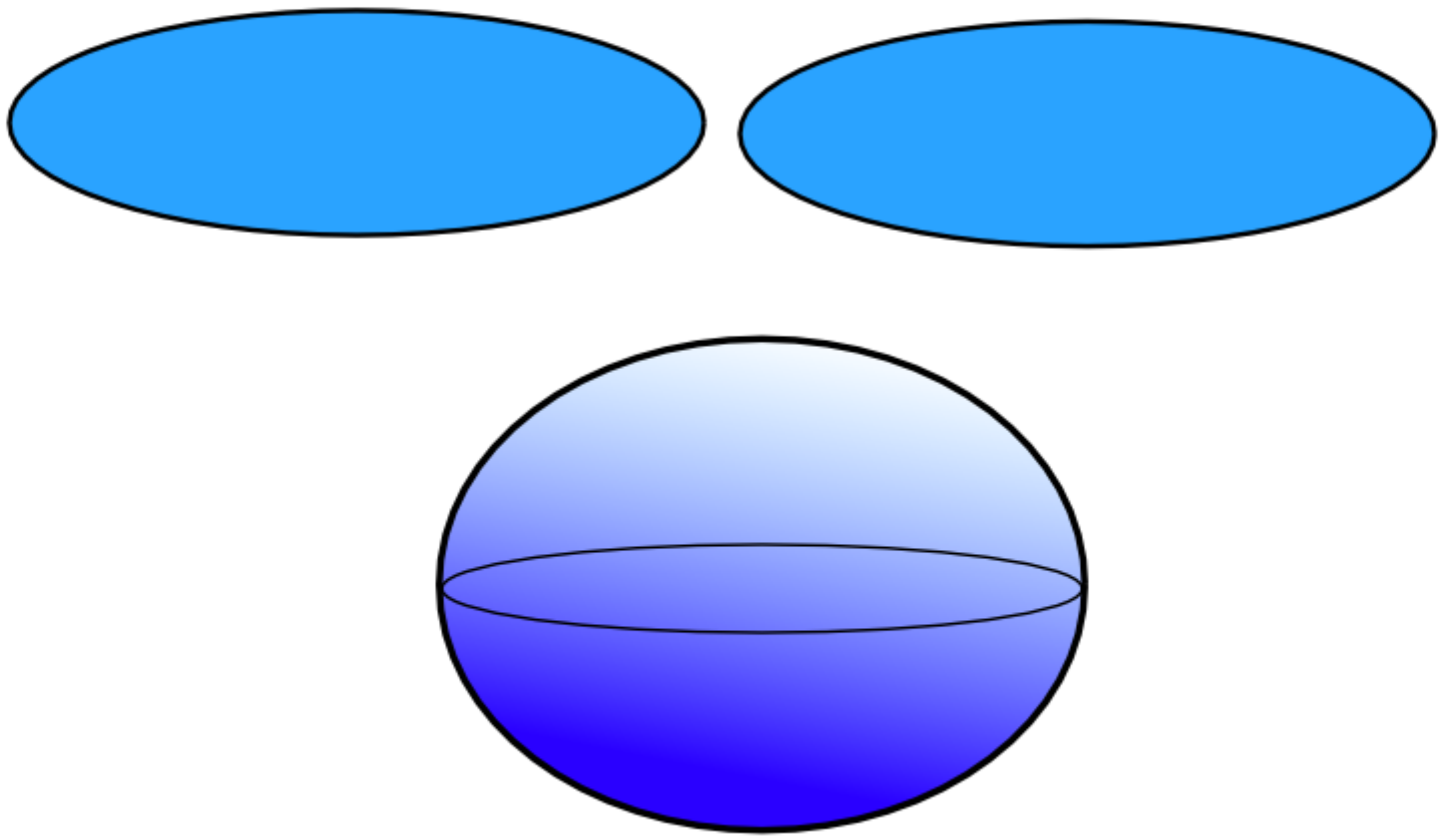
probably the easiest example to think about is the 3-Euclidian space. However, this space is not compact. This space can be made into a compact space, the 3-sphere, by adding one point to it.

However, this method does not tell us how to obtain other 3-manifolds. Is there a better way to think about the 3-sphere?

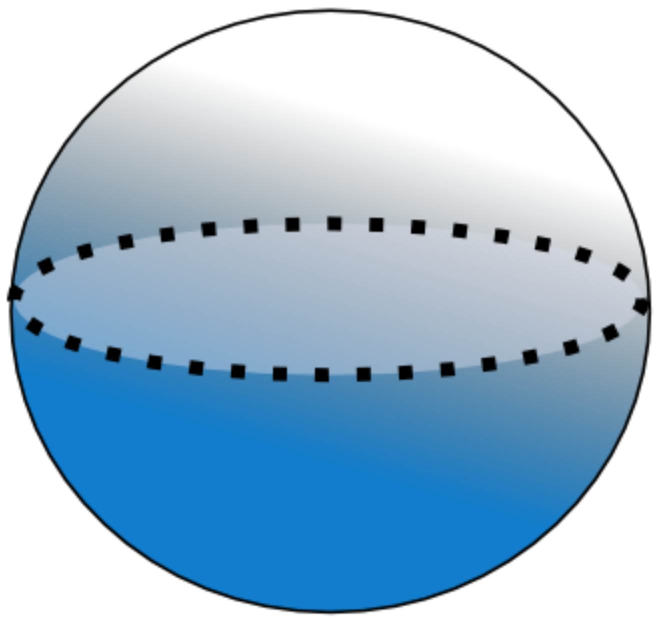
The 2-sphere



The 2-sphere



The 3-sphere



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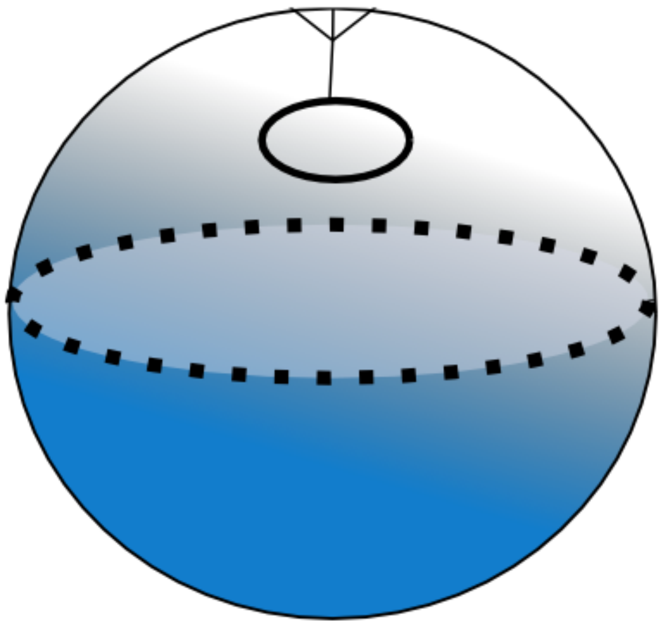

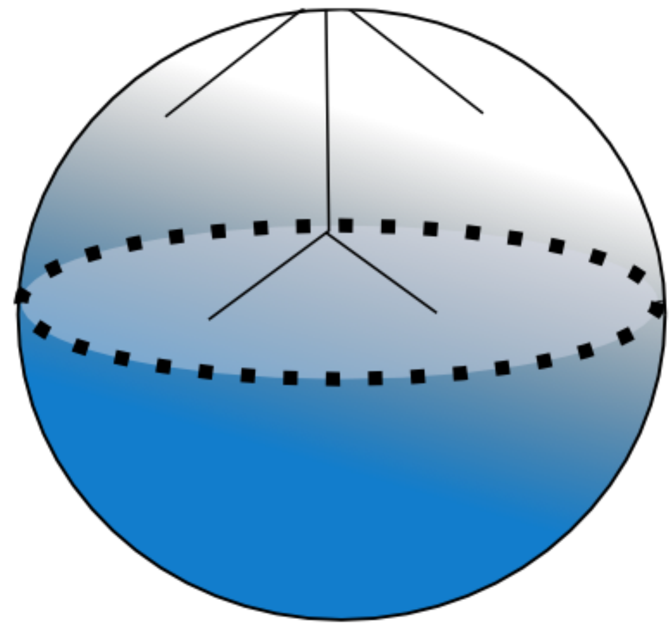

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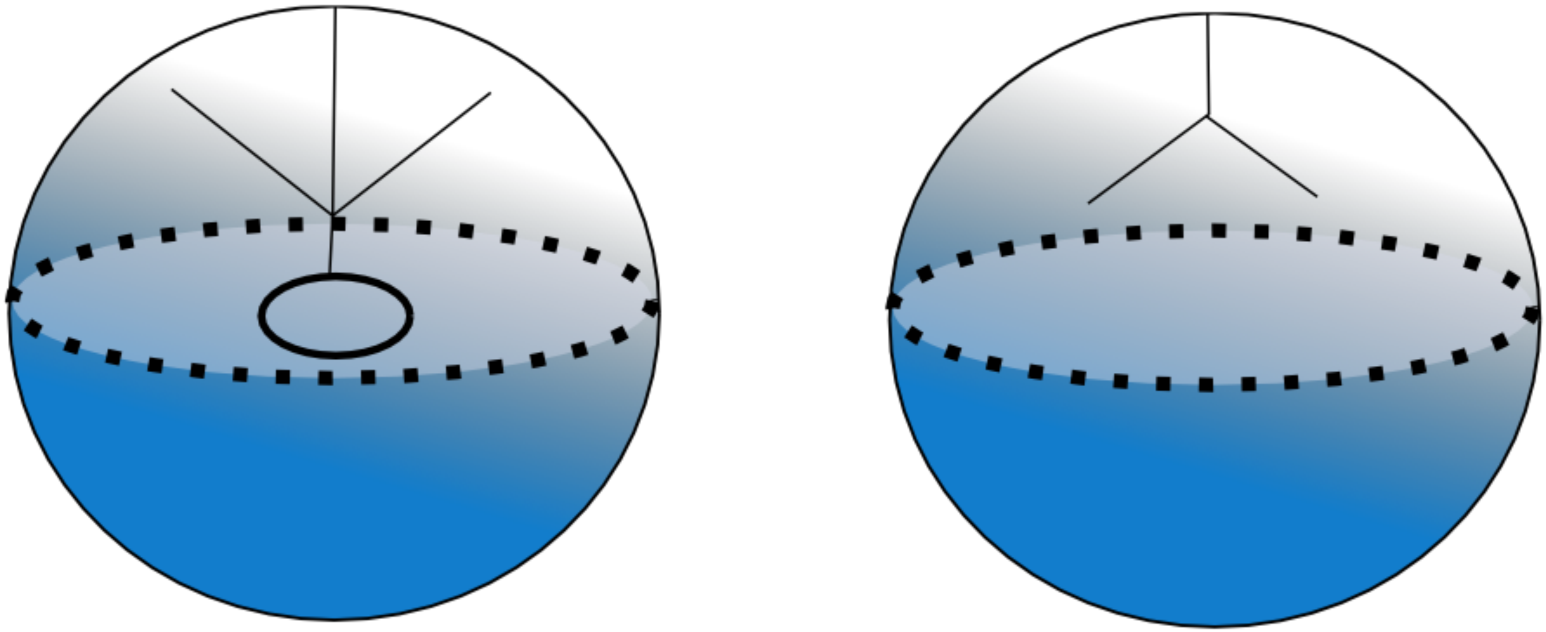

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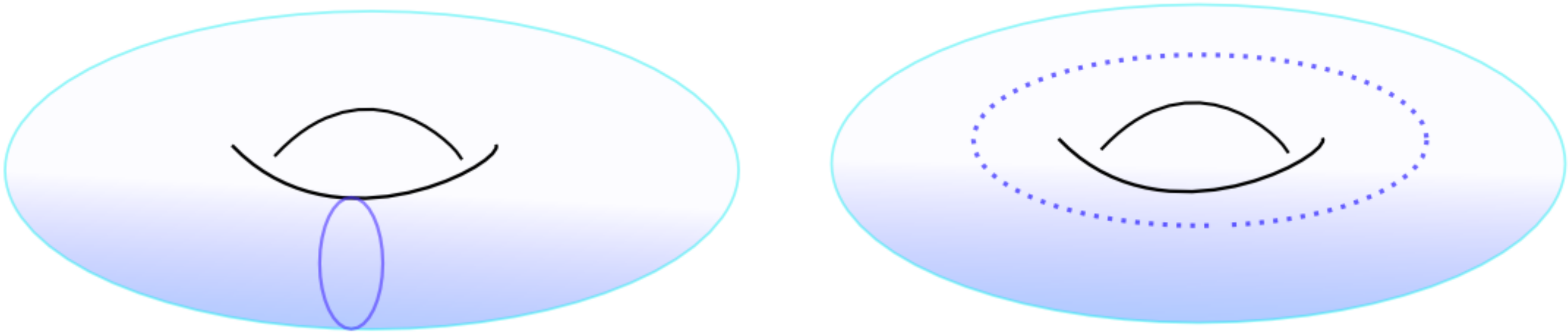

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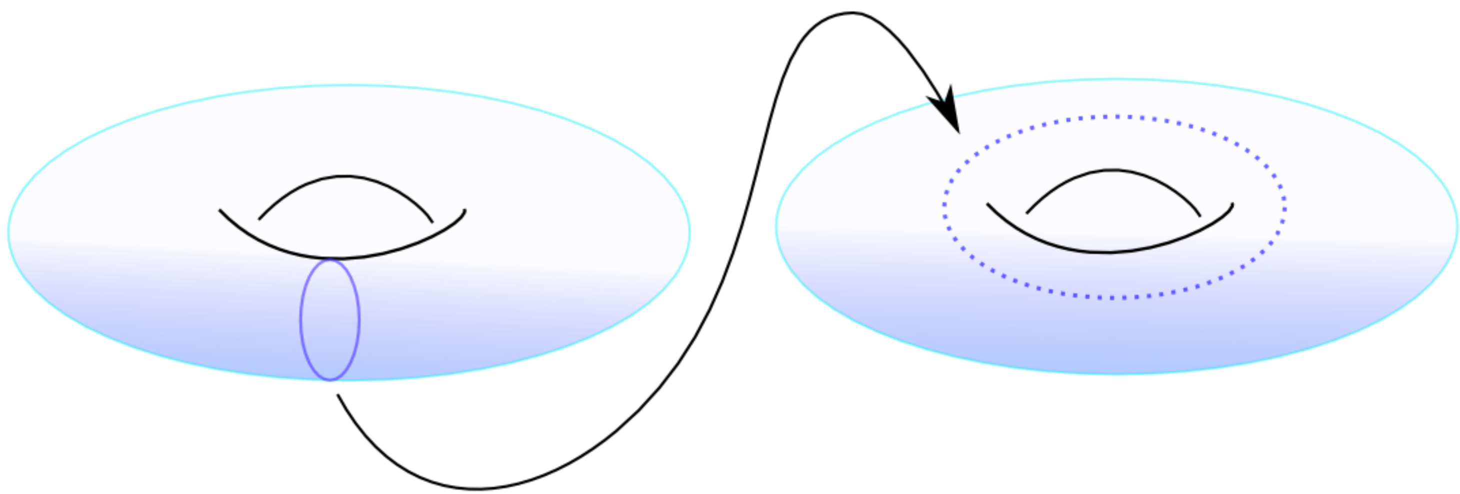

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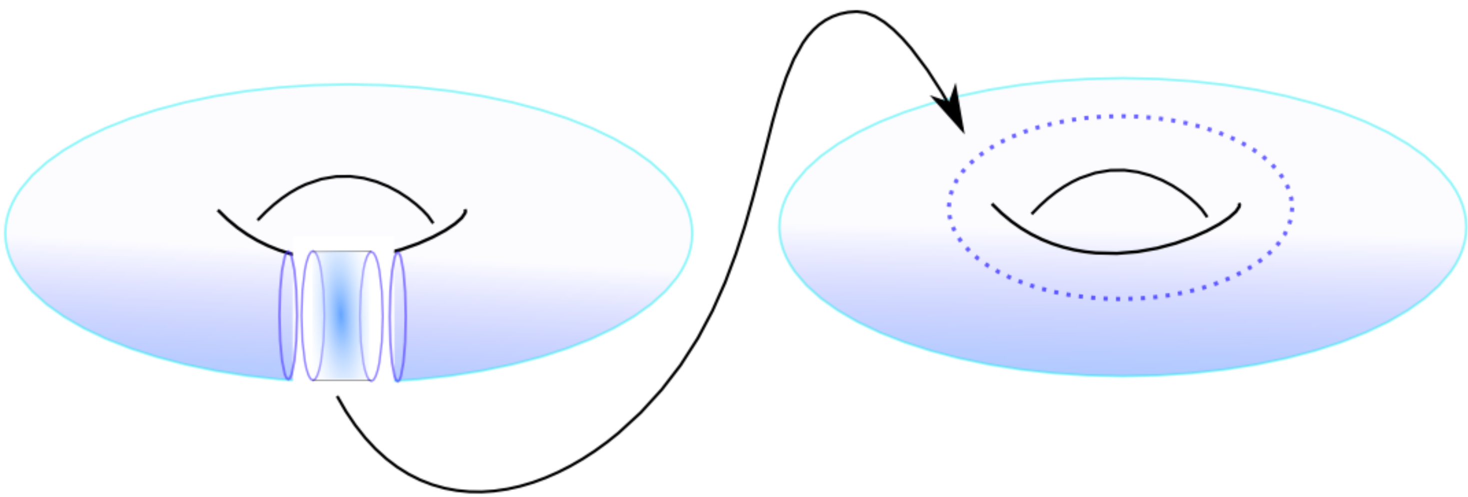

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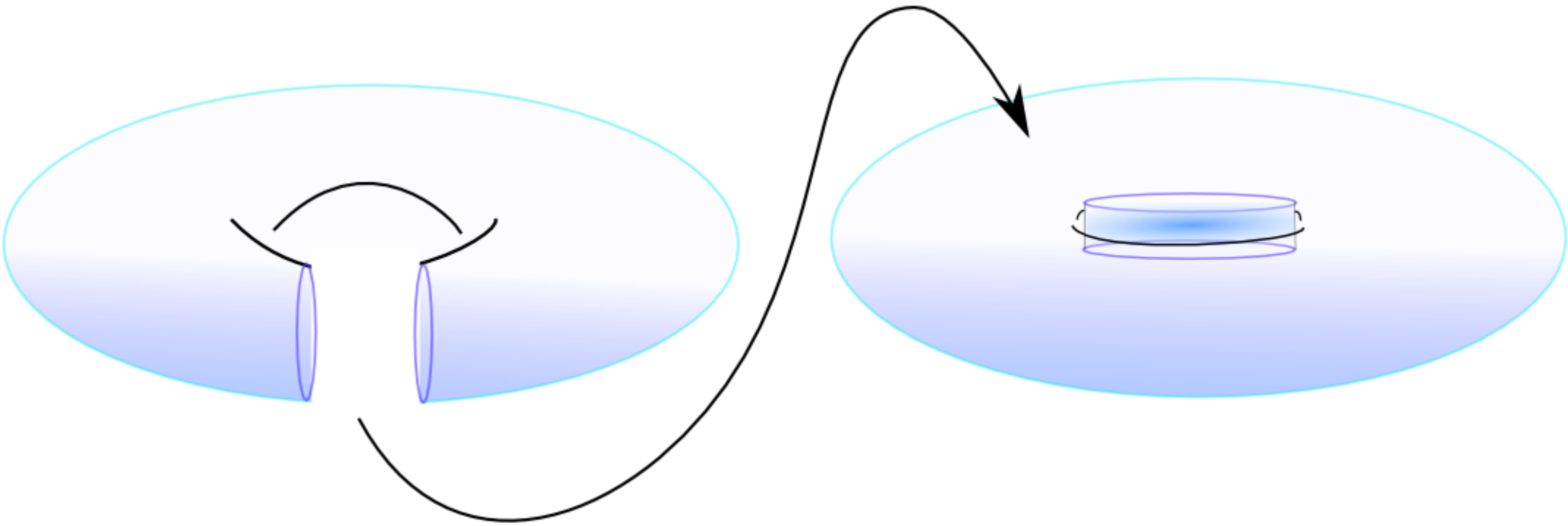

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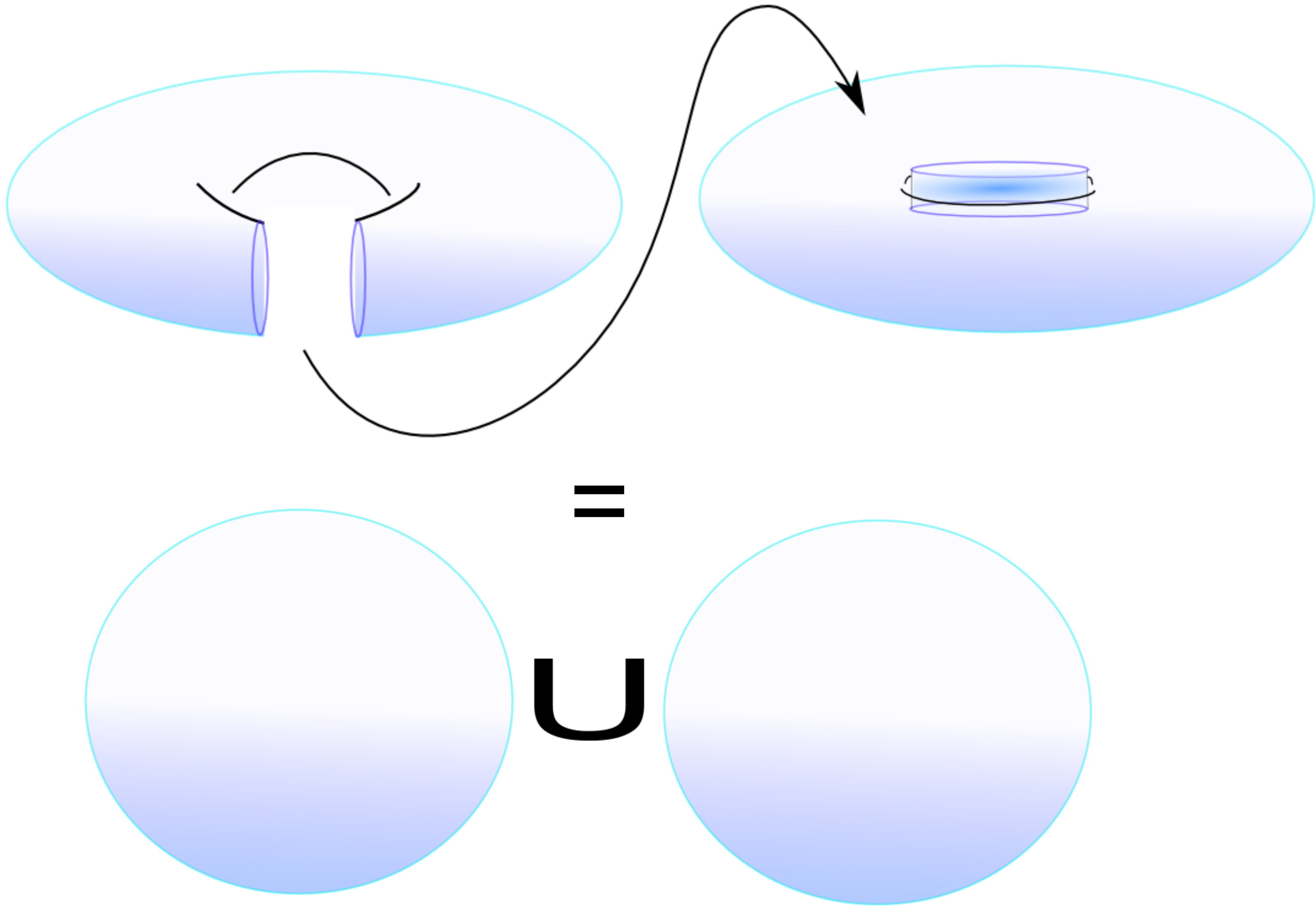

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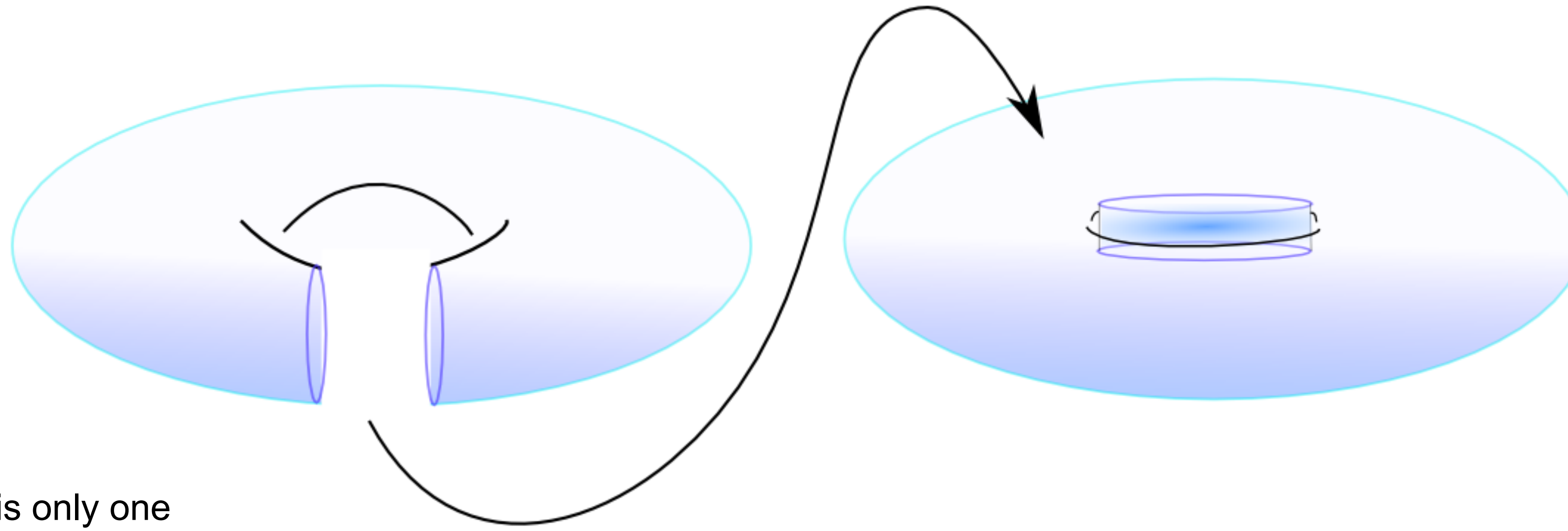

There is only one homeomorphism from the 2-sphere to itself

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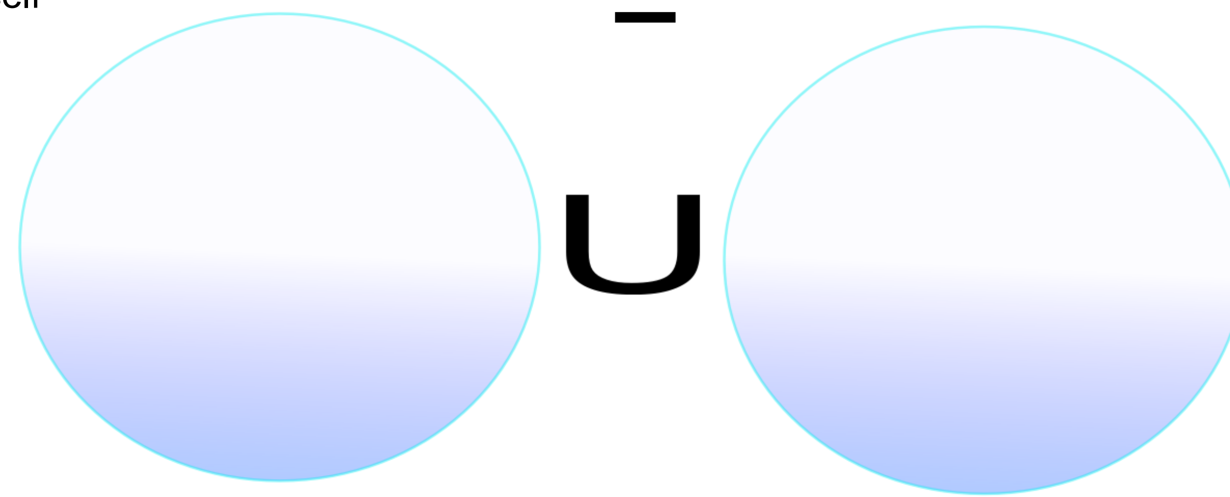
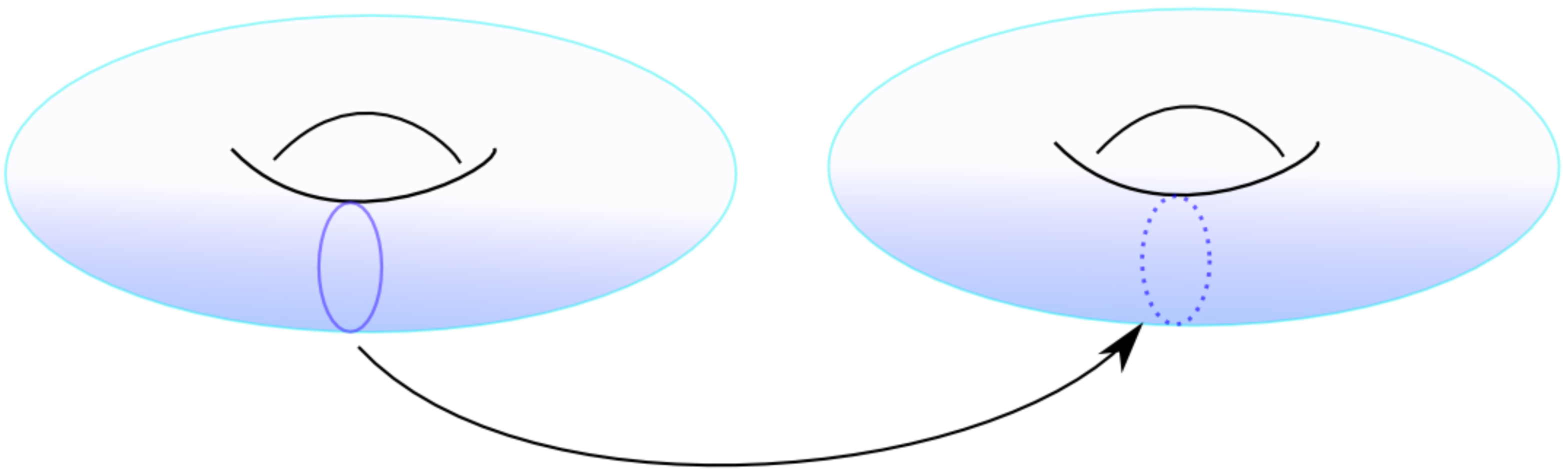


$= S^3$

Gluing a solid torus to a solid torus

This time we map the meridian to the meridian.



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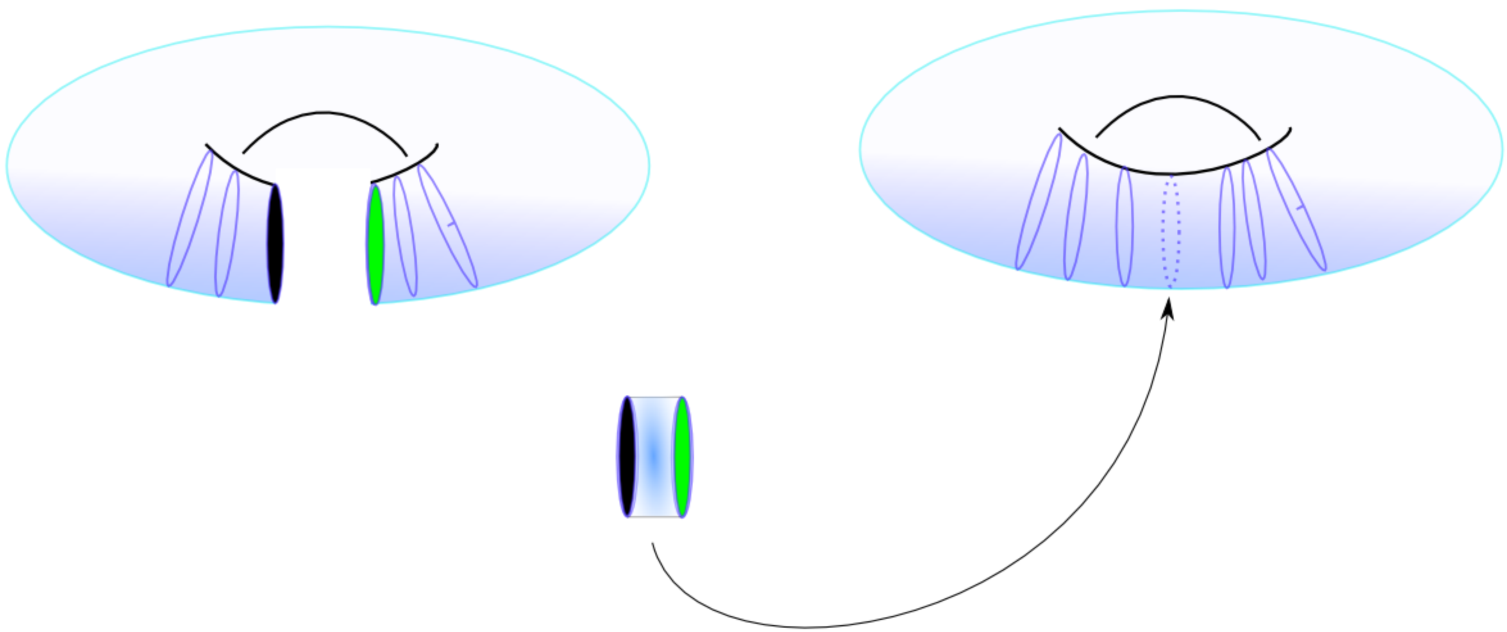

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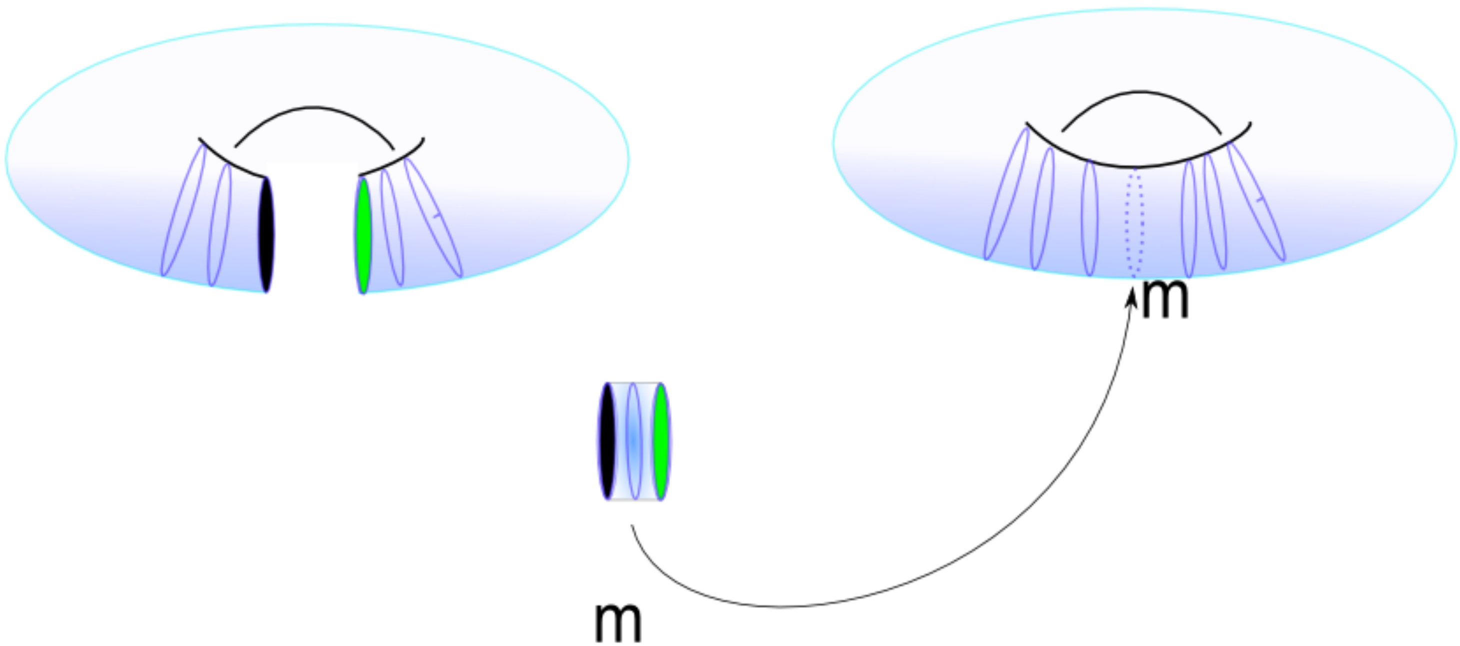

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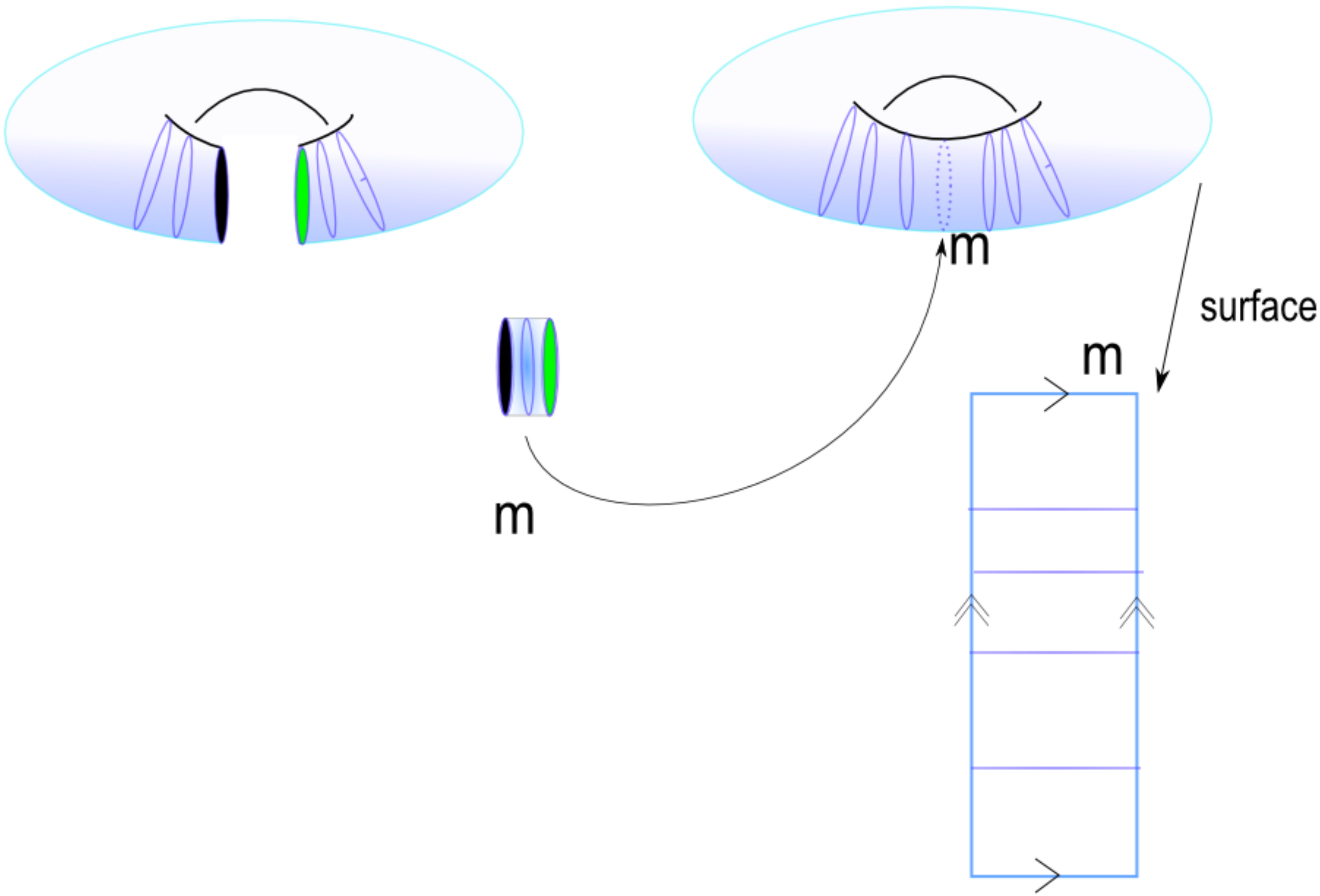

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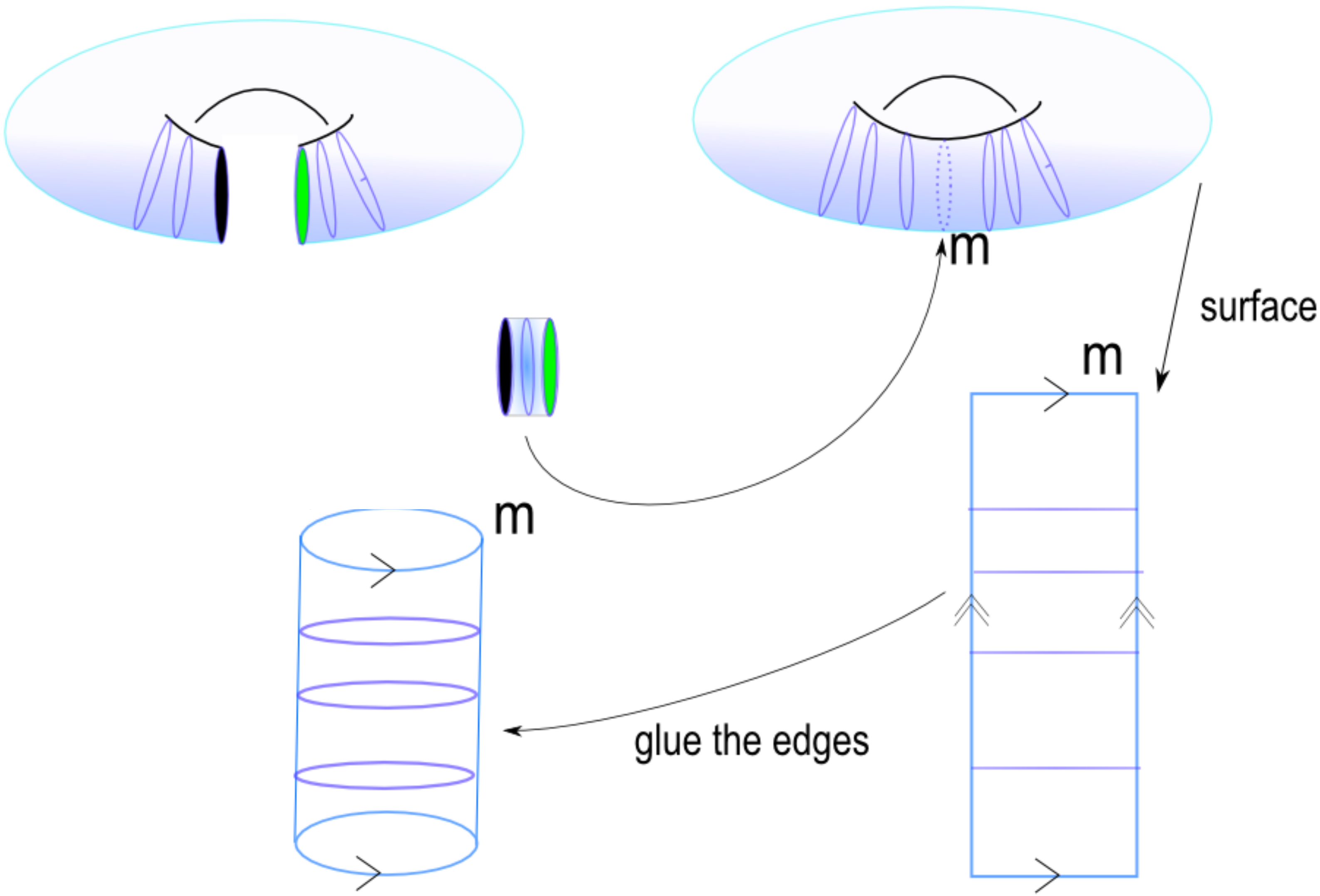

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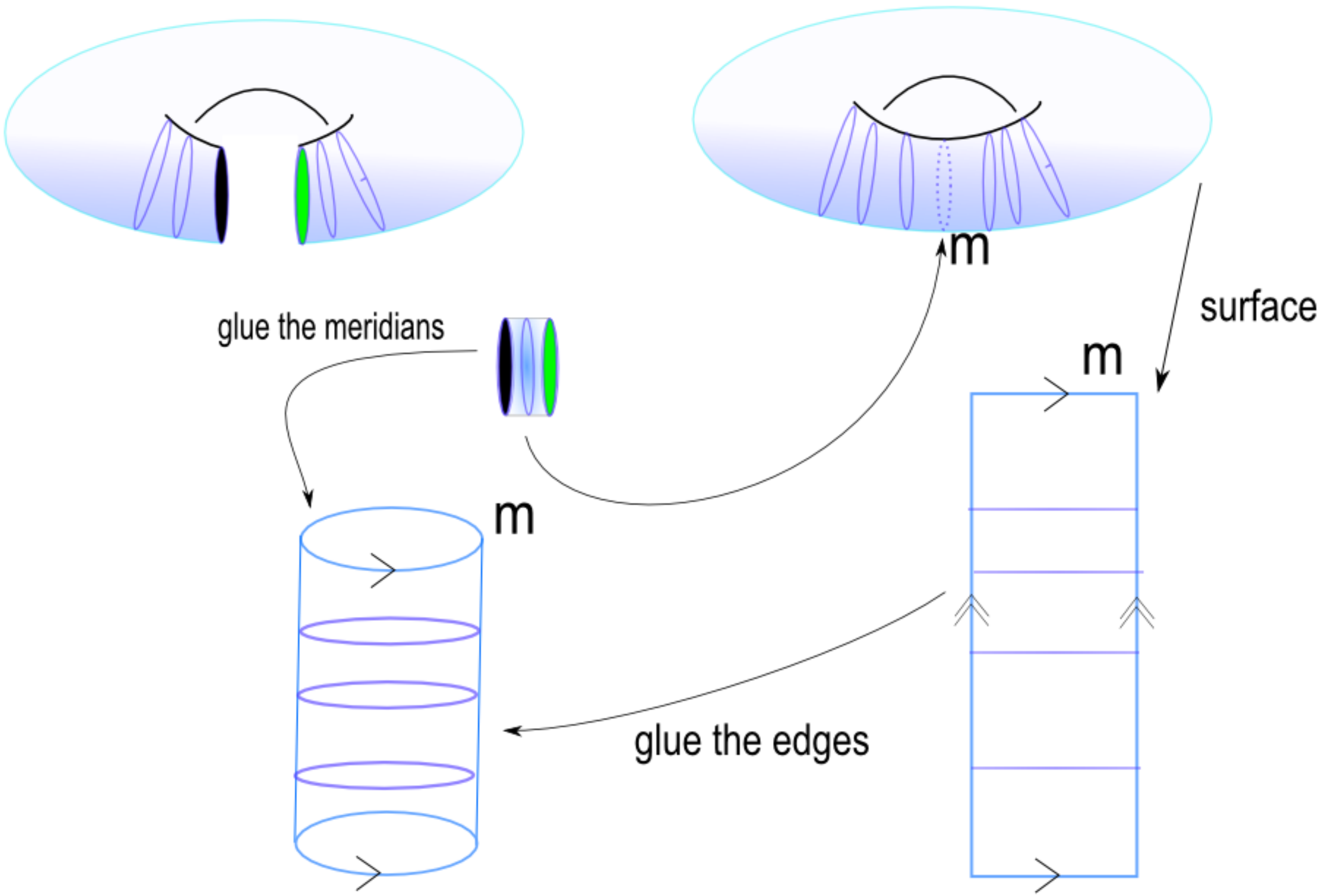

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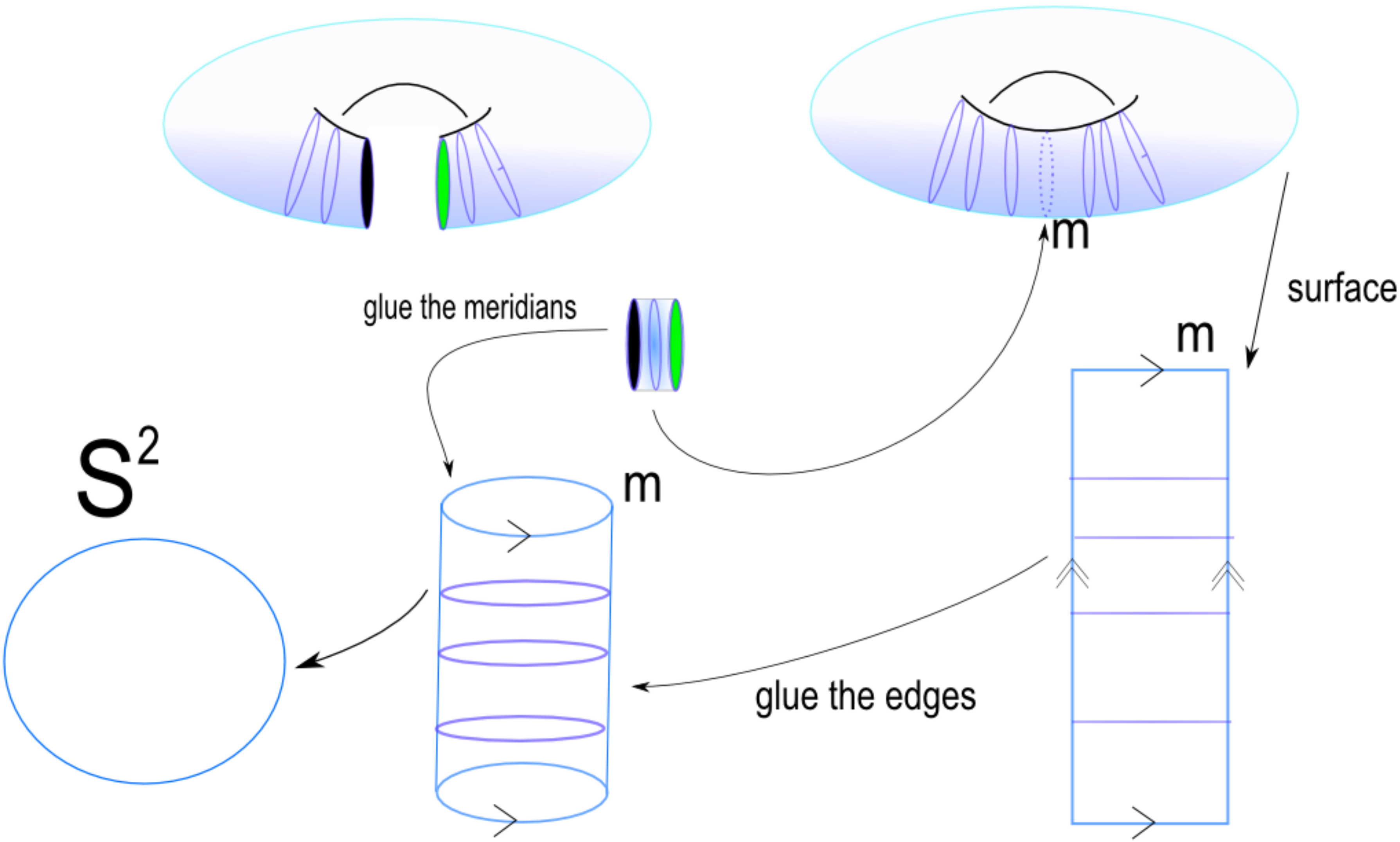

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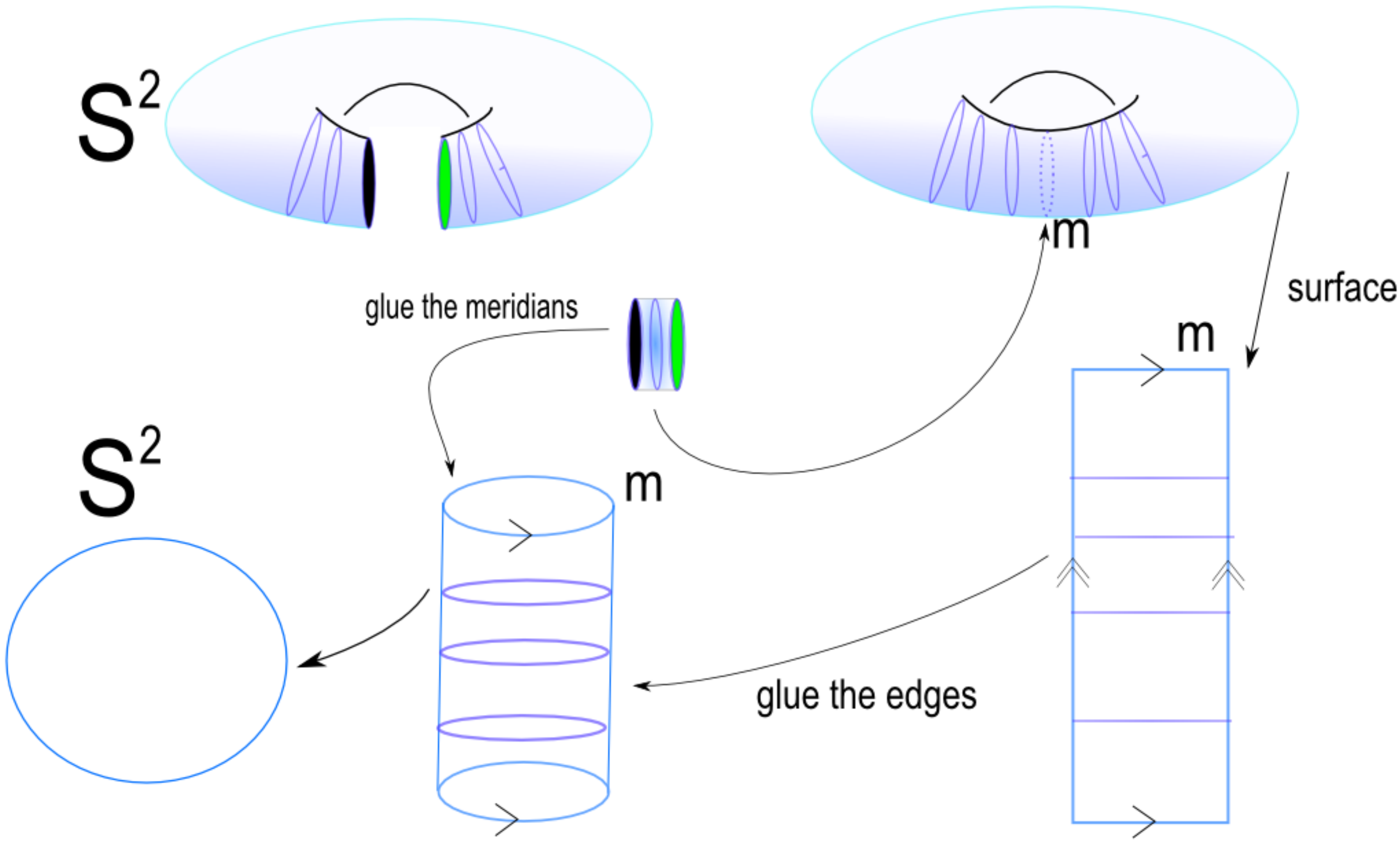
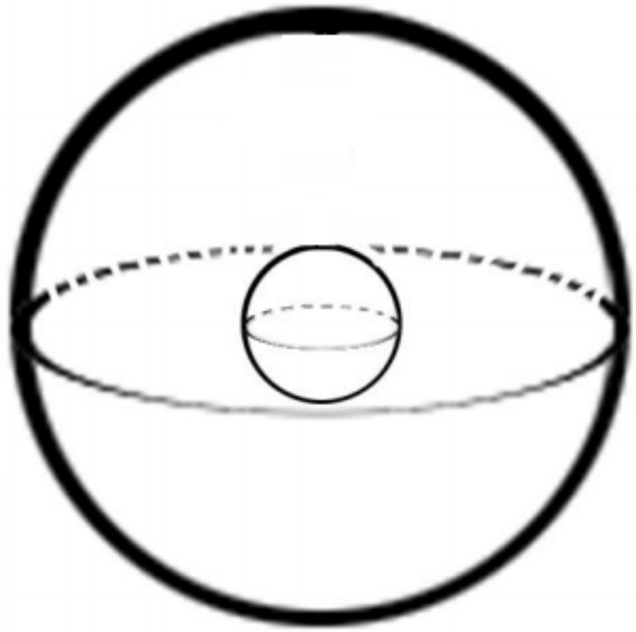


Gluing a solid torus to a solid torus

The 3-manifold that we can obtain by mapping the meridian to a meridian is homeomorphic to S^2 X S^1



Gluing a solid torus to a solid torus

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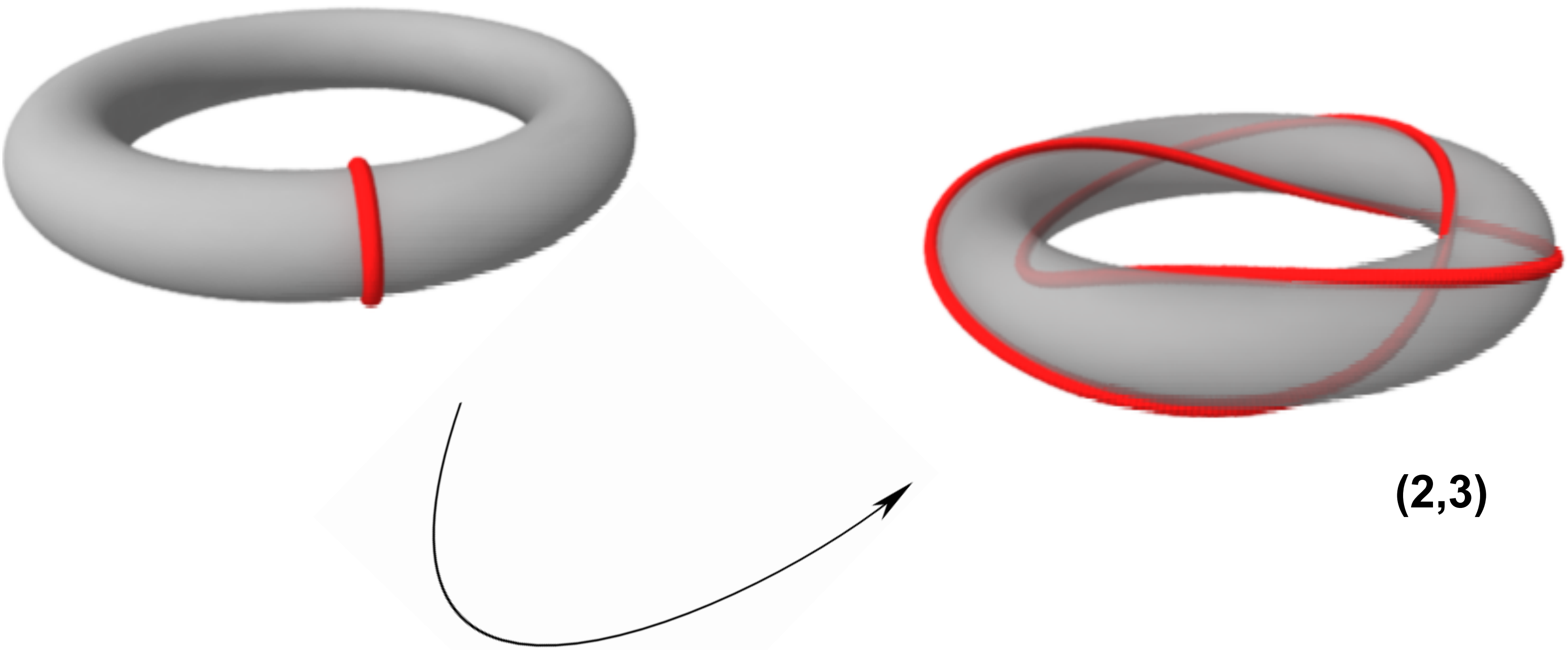

(2,3)

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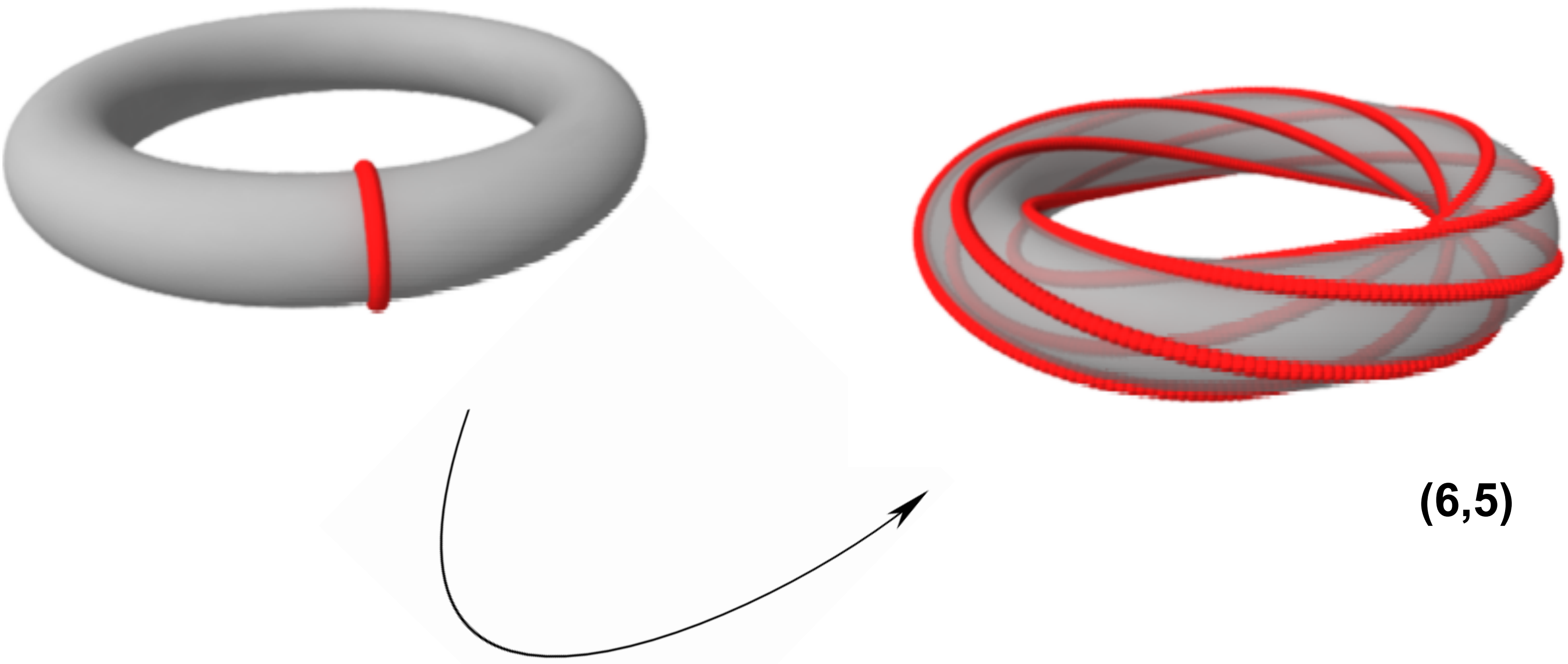

(6,5)

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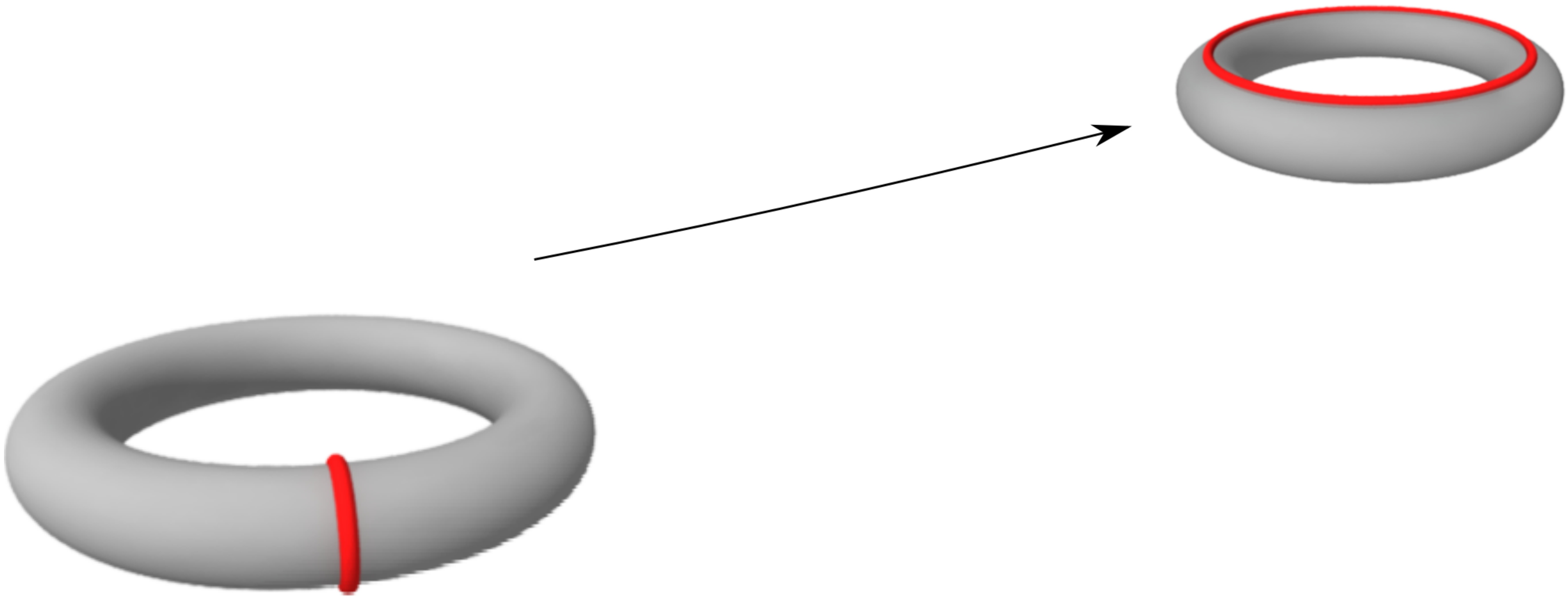

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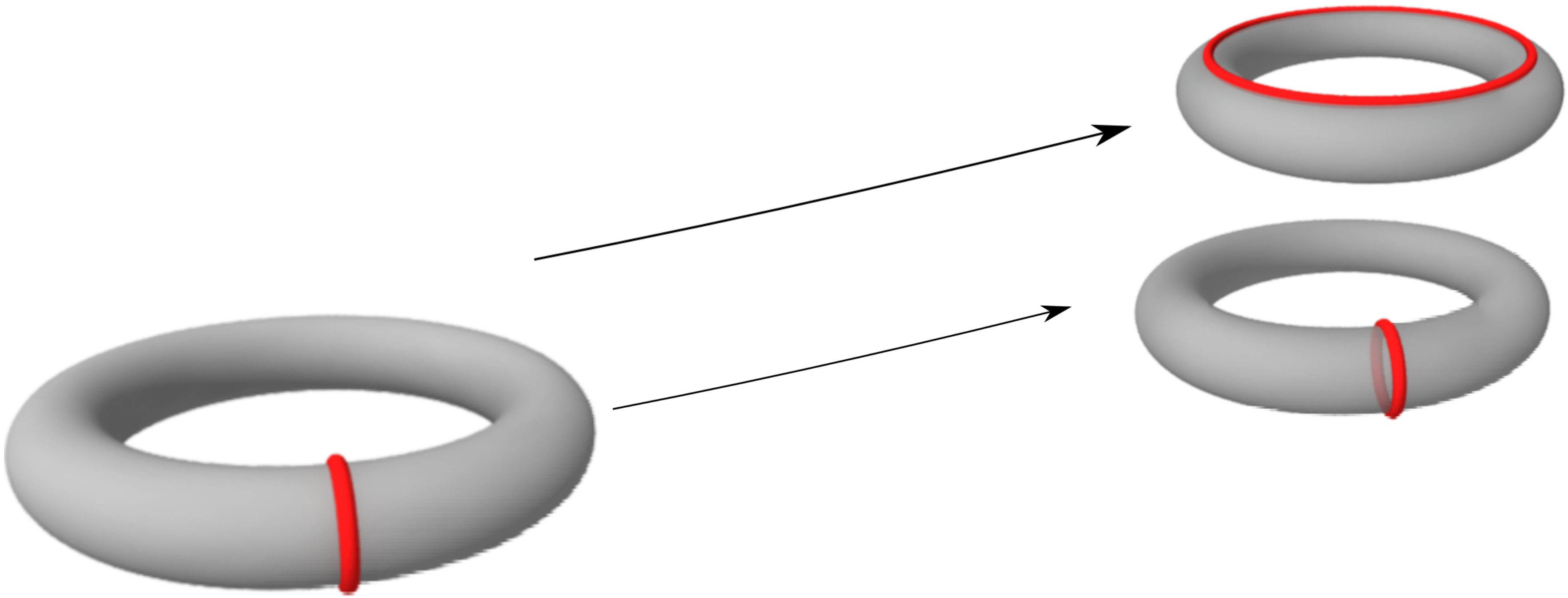

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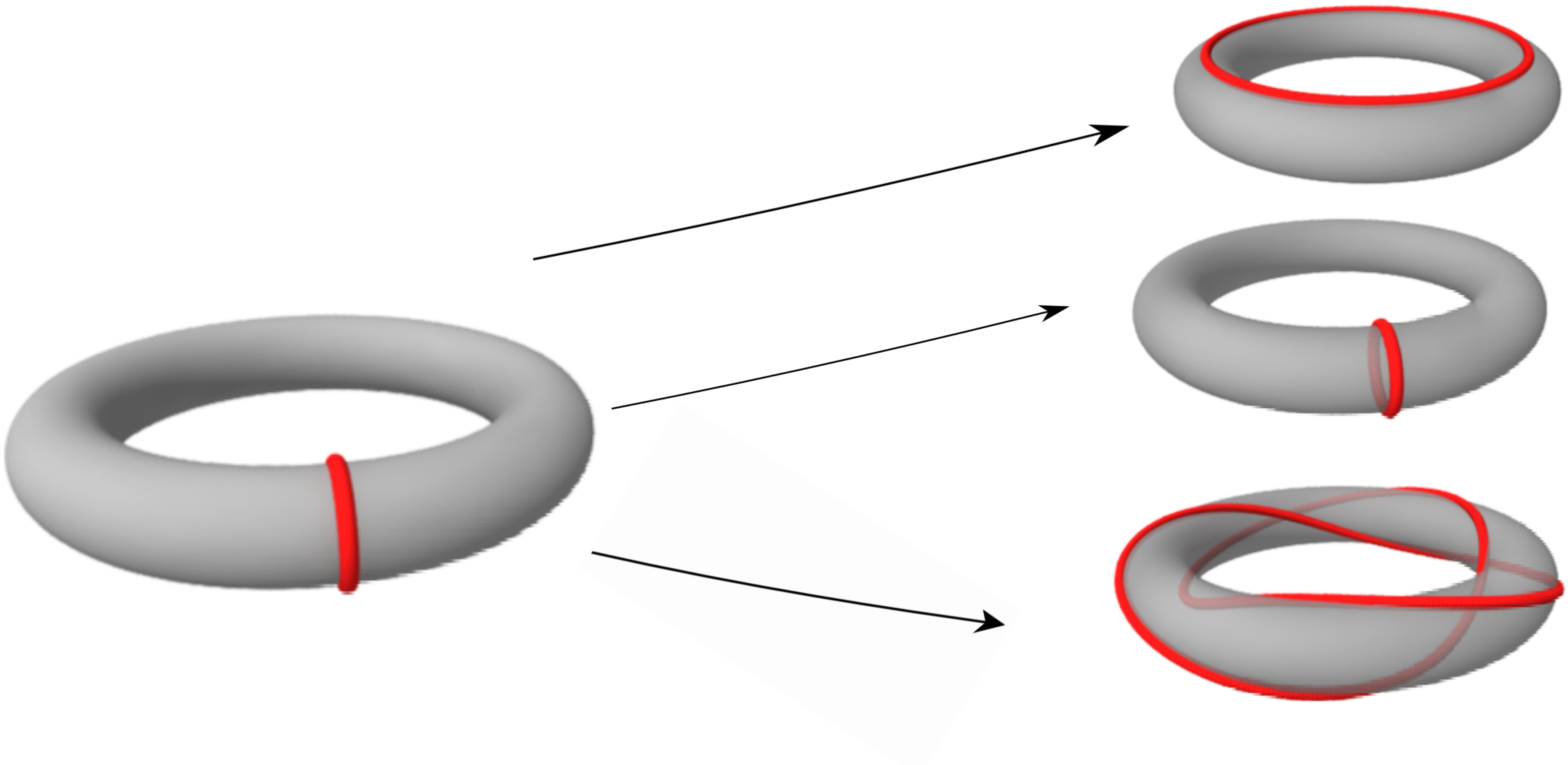

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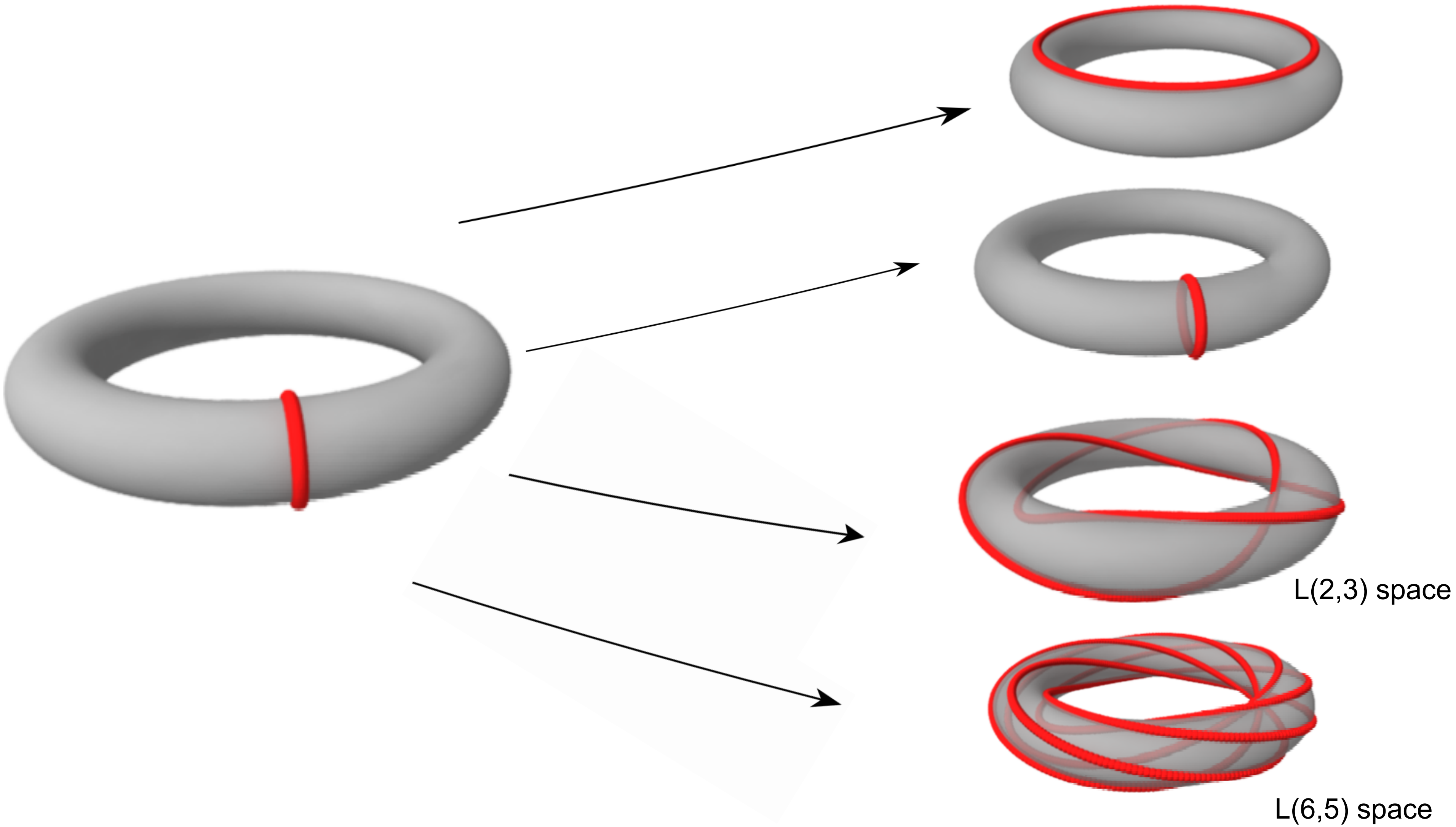

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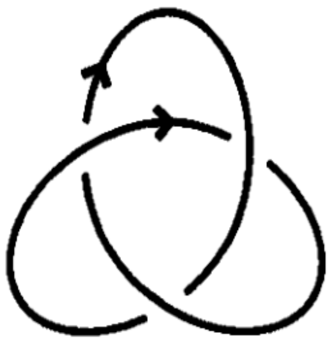
There are 3-manifolds that cannot be realized as L(p,q) manifolds.

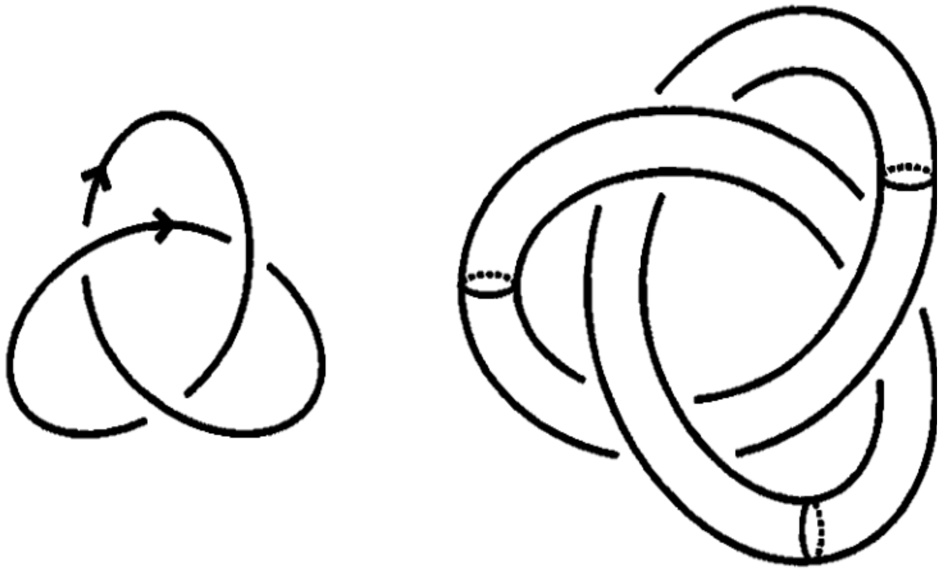
Dehn Surgery

Given a knot K.



Dehn Surgery

Given a knot K. Thicken this knot a little bit. This thickened knot is basically a solid torus. Denote this solid torus V(K).



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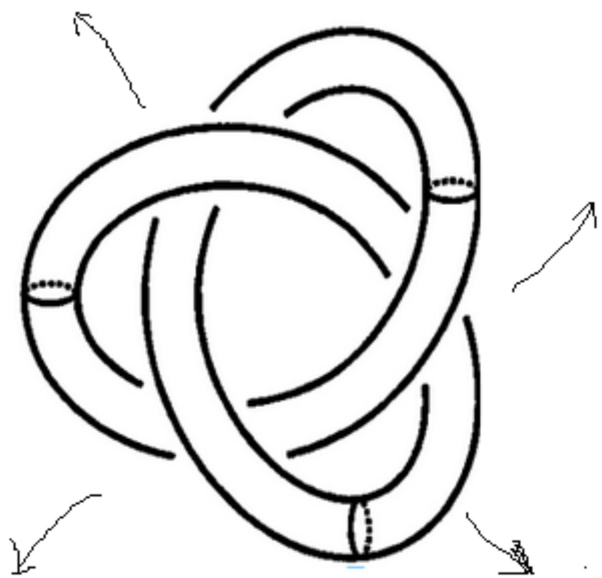

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So now take another solid torus, say ST, and repeat the same operation we did earlier : pick a curve (p,q) on S^3\V(K) and then map the meridian of ST to the curve (p,q).

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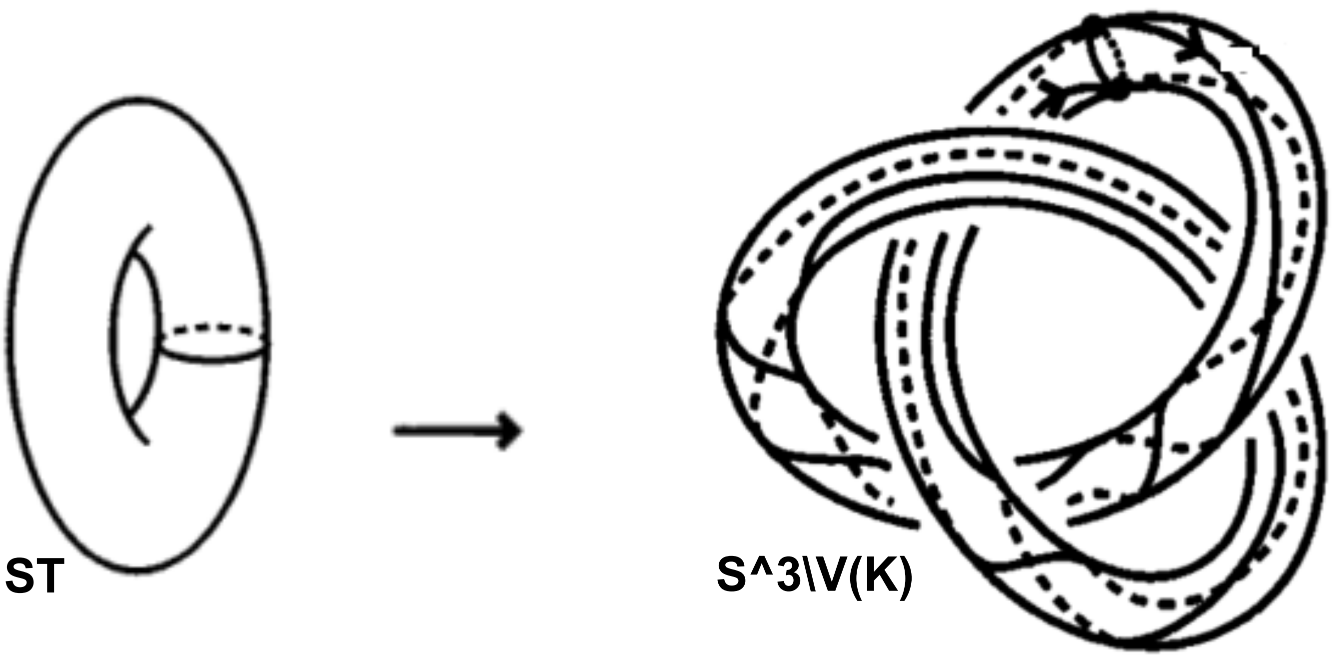

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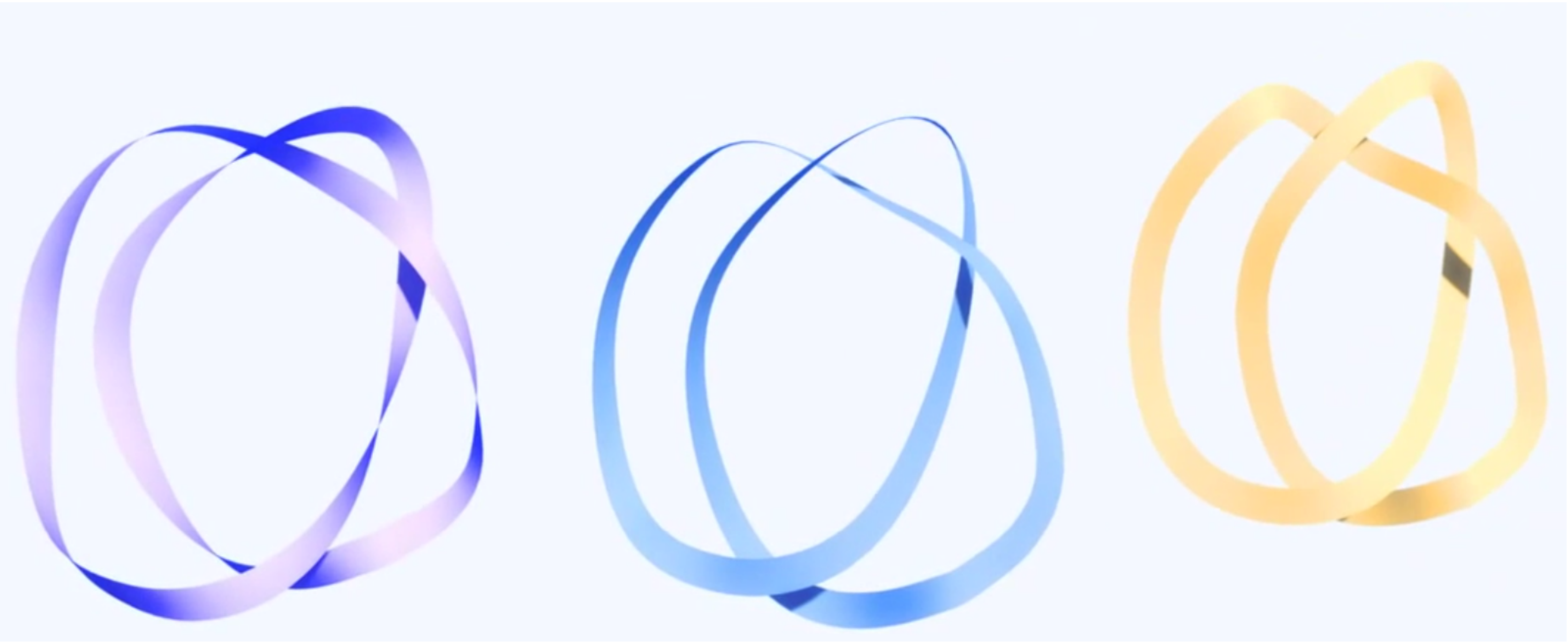
All closed, oriented 3-manifolds can be obtained by doing a Dehn surgery on some link.

Dehn surgery requires specifying a knot and a curve (p,q) (p and q are coprime).

Can we generate 3-manifolds without using the “data” (p,q)?

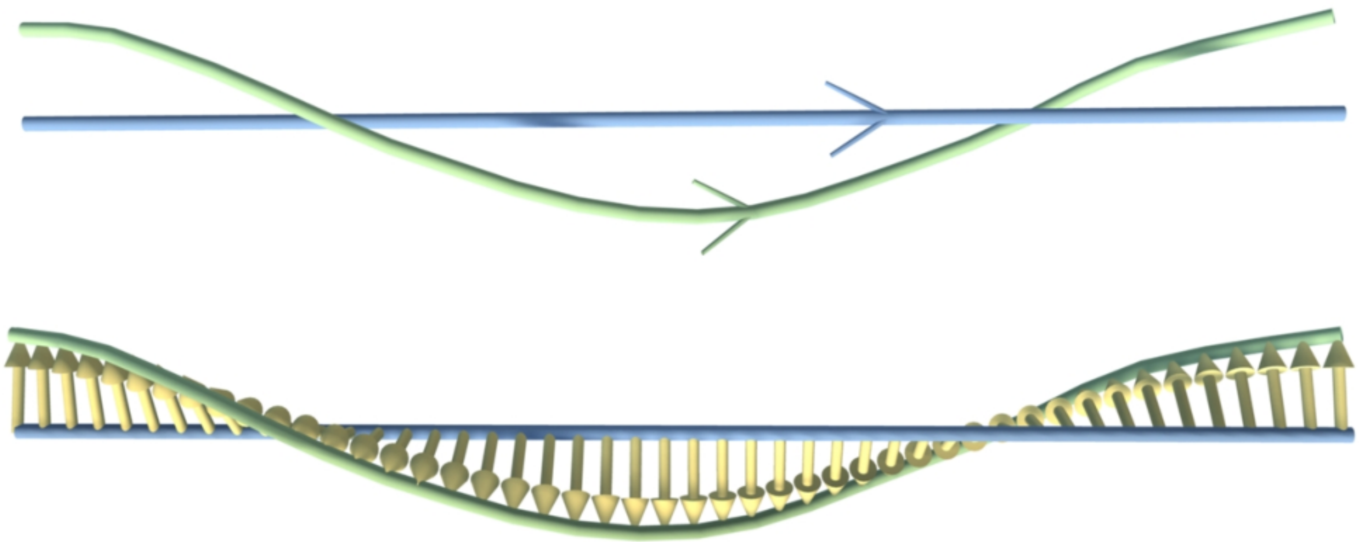
Framed knots

Framed knots



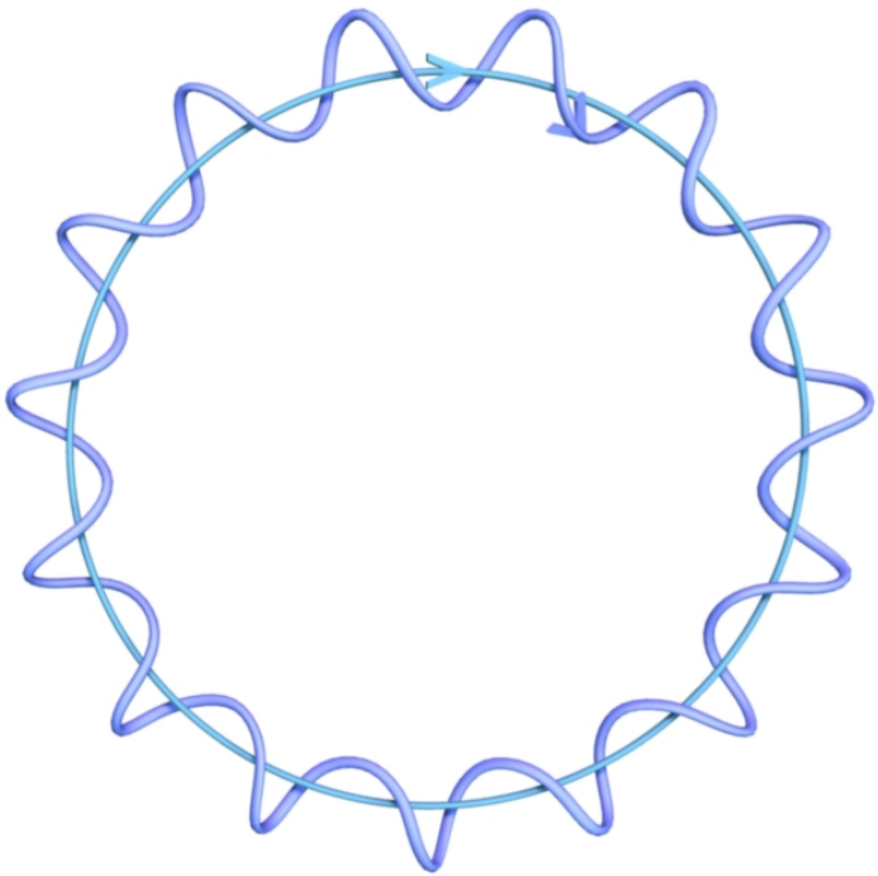
Framed knots

one can think about the framing as a curve that runs parallel to the knot



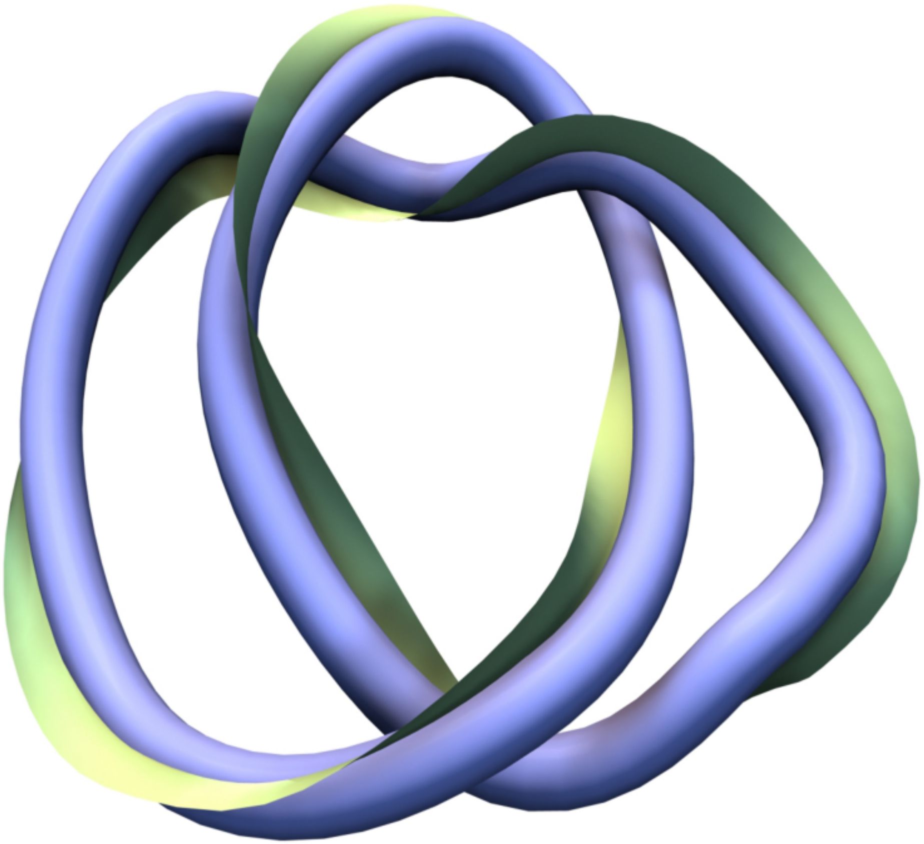
Framed knots

For example, the following framing of the trivial knot



Framed knots

Another way to think about framing: it is a curve on the torus that goes around the knot. It other words, a framing determines a curve of the torus boundary.



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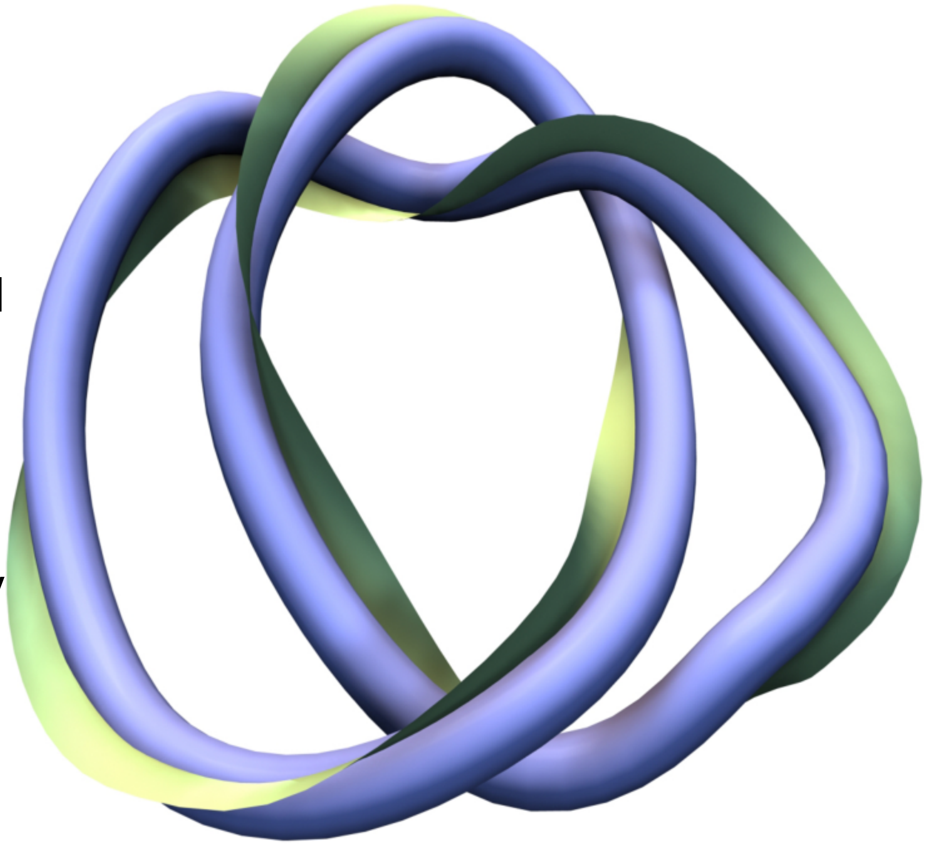
This curve is obtained by rotating p times around meridian and 1 time around the longitude.

The Dehn surgery that one obtain using this curve is called an integral surgery



Framed knots and 3-manifolds

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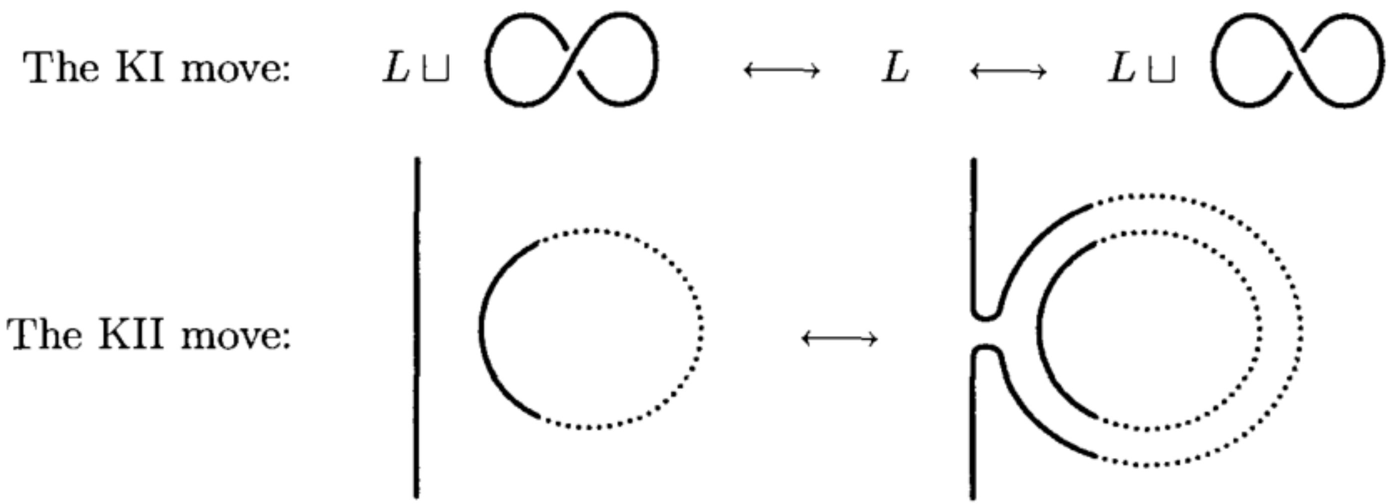
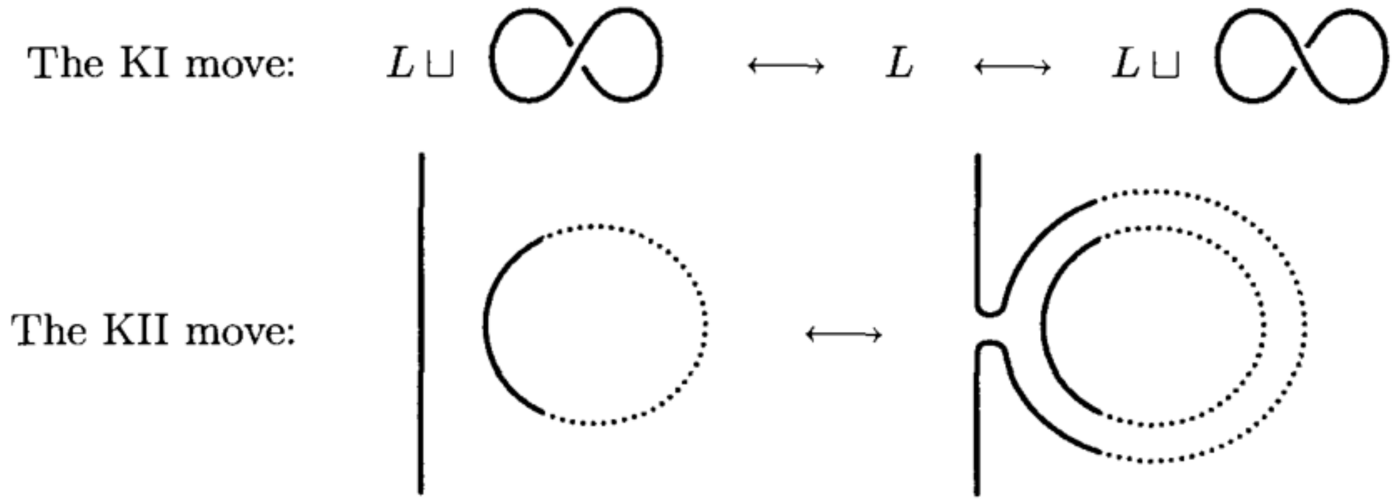
Theorem (Kirby) For framed links L and L' in S^3, the 3-manifold S^3_L is homeomorphic to $S^3_{L'}$ if and only if L can be obtained from L' by a sequence of isotopies of framed links and the KI and KII moves.

Kirby moves



Using these 2 theorems, if we define a quantity on framed knots that does not change under isotopy of framed links and Kirby moves, one obtains an invariant of 3-manifolds.

Kirby moves



Using these 2 theorems, if we define a quantity on framed knots that does not change under isotopy of framed links and Kirby moves, one obtains an invariant of 3-manifolds.

This method has been used effectively to distinguish between 3-manifolds.

Thank You