The Colored Jones Polynomial

Junior Topology Seminar
Mustafa Hajiij

Note: Some figures in this file have been taken from the paper : "An Introduction to the Volume Conjecture" by Hitoshi Murakami

Alexander's Theorem

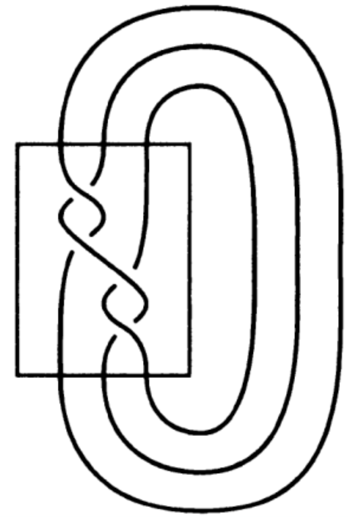
Any tame oriented link L in 3-space may be represented by a pair (b, n), where b is an element of the n-string braid group B_n.

The link *L* is obtained by closing *b*

b

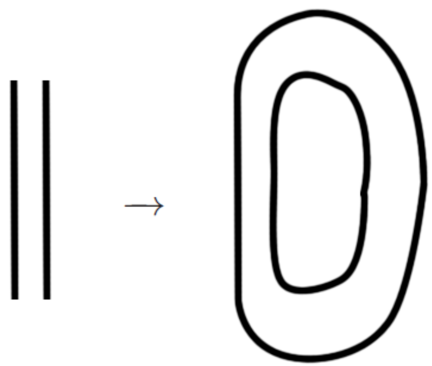
$\rightarrow$

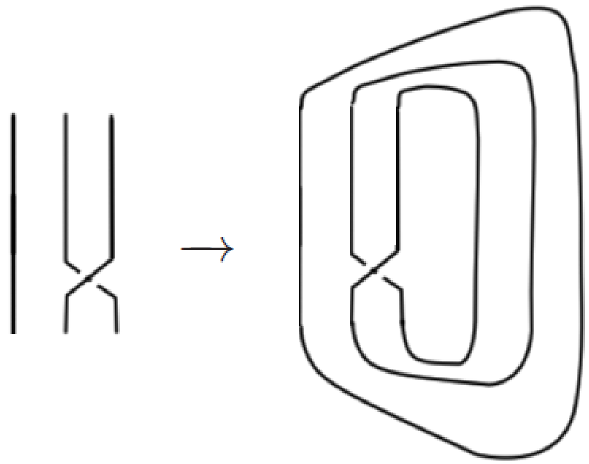
L



Alexander's Theorem

The problem with Alexander's theorem is that many differnet braids can give us the same knot.

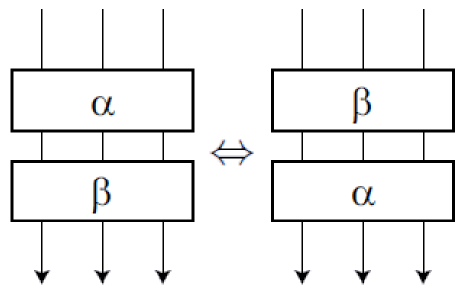




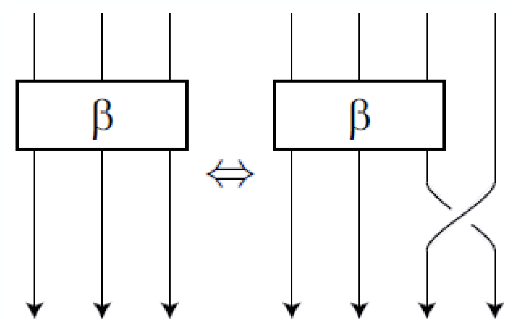
Markov's theorem

b and b' give equivalent links $\Longleftrightarrow$ b and b' are related by

conjugation



stabilization



Link invariants

Alexander's and Markov's theorems can be used to define link invariants.

The enhanced Yang-Baxter operator

V: a N-dimensional vector space over $\mathbb{C}$.

$R\colon V \otimes V \to V \otimes V$ (R-matrix), $\mu\colon V \to V$: isomorphisms

a, b: non-zero complex numbers.

The enhanced Yang-Baxter operator

V: a N-dimensional vector space over $\mathbb{C}$.

$R\colon V \otimes V \to V \otimes V$ (R-matrix), $\mu\colon V \to V$: isomorphisms

a, b: non-zero complex numbers.

A quadruple (R, μ, a, b) is called an enhanced Yang–Baxter operator if it satisfies the following:

(1) $(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)$,

(2) $R(\mu \otimes \mu) = (\mu \otimes \mu)R$,

(3) $\mathrm{Tr}_2\big(R^{\pm 1}(\mathrm{Id}_V \otimes \mu)\big) = a^{\pm 1} b \, \mathrm{Id}_V$.

The enhanced Yang-Baxter operator

Here $\mathrm{Tr}_k\colon \mathrm{End}(V^{\otimes k}) \to \mathrm{End}(V^{\otimes(k-1)})$ is defined by

$$\mathrm{Tr}_k(f)(e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_{k-1}}) := \sum_{j_1, j_2, \ldots, j_{k-1}, j=0}^{N-1} f^{j_1, j_2, \ldots, j_{k-1}, j}_{i_1, i_2, \ldots, i_{k-1}, j} (e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_{k-1}} \otimes e_j)$$

where $f \in \mathrm{End}(V^{\otimes k})$ is given by

$$f(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1, j_2, \ldots, j_k=0}^{N-1} f^{j_1, j_2, \ldots, j_k}_{i_1, i_2, \ldots, i_k} (e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k})$$

and $\{e_0, e_1, \ldots, e_{N-1}\}$ is a basis of V.

A representation for the braid group

We will define a representation for the braid group.

A representation for the braid group

We will define a representation for the braid group.

$$B_n \to \mathrm{Aut}(V^{\otimes n})$$

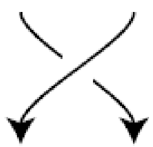
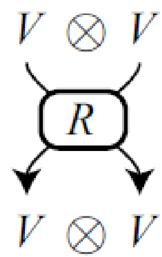
σ_i into R_i for $i = 1, \ldots, n-1$

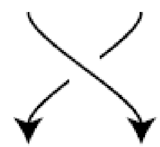
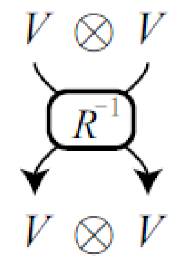
A representation for the braid group

We will define a representation for the braid group.

$$B_n \to \mathrm{Aut}(V^{\otimes n})$$

$$\sigma_i \text{ into } R_i \text{ for } i = 1, \ldots, n-1$$

Replace  with 

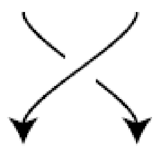
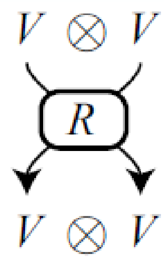
, and  with 

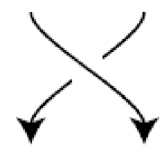
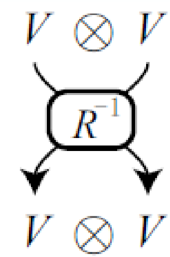
A representation for the braid group

We will define a representation for the braid group.

$$B_n \to \mathrm{Aut}(V^{\otimes n})$$

$$\sigma_i \text{ into } R_i \text{ for } i=1, \ldots, n-1$$

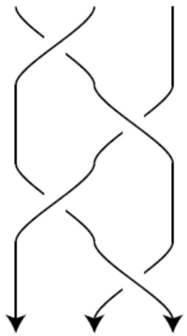
Replace  with 

, and  with 

n-braid $\beta \Rightarrow$ homomorphism $\Phi(\beta):\ V^{\otimes n} \longrightarrow V^{\otimes n}$

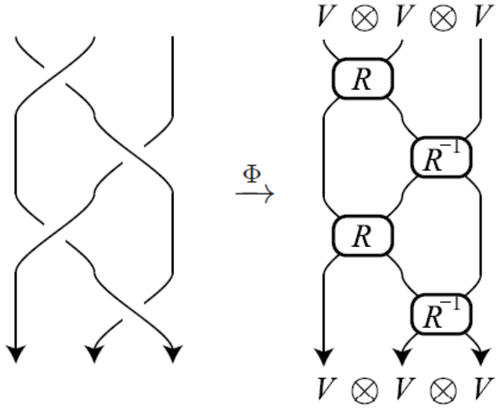
Example

$$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$$



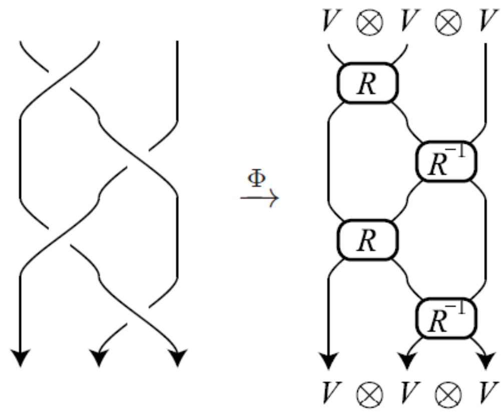
Example

$\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$



Example

$$\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$$



$$\Phi(\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}) = (R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})$$

Trace invariant

n-braid $\beta \Rightarrow$ a link L.

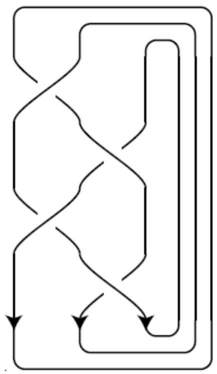
$$T_{(R,\mu,a,b)}(L) := a^{-w(\beta)} b^{-n}\, \mathrm{Tr}_1\Big(\mathrm{Tr}_2(\cdots(\mathrm{Tr}_n\,(\Phi(\beta)\mu^{\otimes n}))\cdots)\Big)$$

Example

$$\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$$

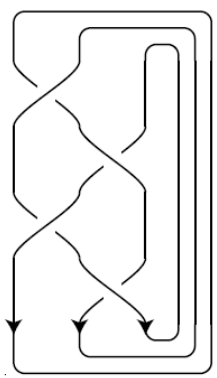
Example

$$\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \Rightarrow$$

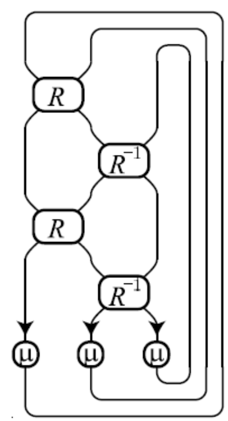


Example

$$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} \;\Rightarrow$$

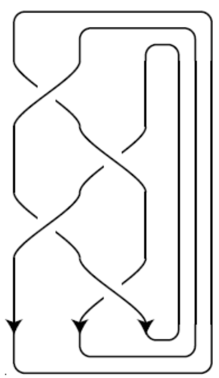


$\Rightarrow$

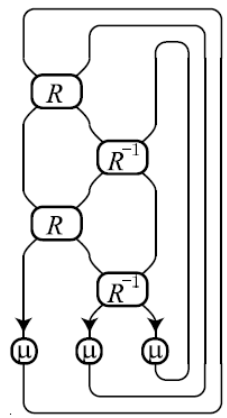


Example

$$\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \Rightarrow$$



$\Rightarrow$



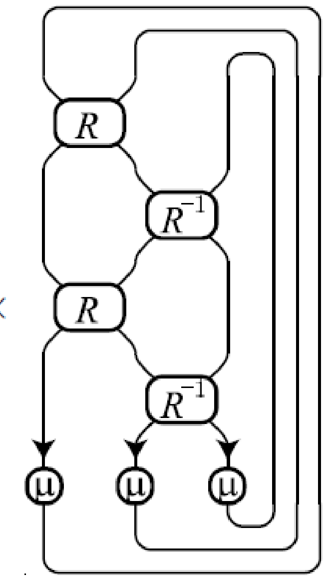
$$\Rightarrow (R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu)$$

Example

$$(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu)$$

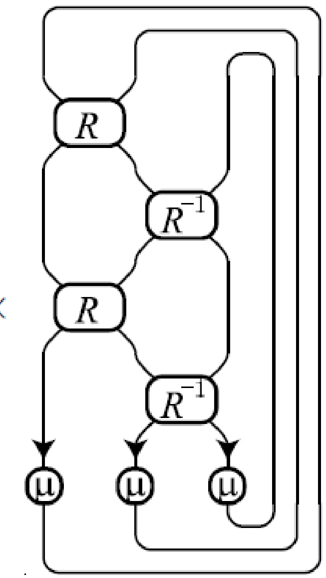
Example

$$(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu) \quad \Rightarrow \quad a^{-w(\beta)} b^{-n} \times$$



Example

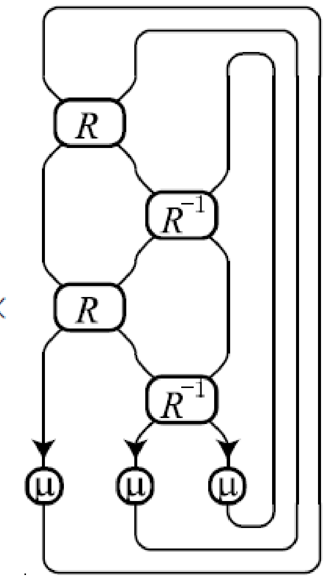
$$(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu) \quad \Rightarrow \quad a^{-w(\beta)} b^{-n} \times$$



$$w(\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}) = +1 - 1 + 1 - 1 = 0$$

Example

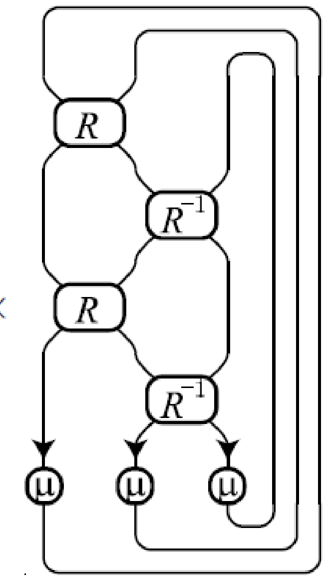
$$(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu) \quad \Rightarrow \quad a^{-w(\beta)} b^{-n} \times$$



$$\Rightarrow \quad b^{-2}\,\mathrm{Tr}_1\big(\mathrm{Tr}_2(\mathrm{Tr}_3((R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu)))\big)$$

Example

$$(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu) \quad \Rightarrow \quad a^{-w(\beta)} b^{-n} \times$$



$$\Rightarrow \quad b^{-2}\,\mathrm{Tr}_1\big(\mathrm{Tr}_2(\mathrm{Tr}_3((R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu)))\big)$$

And this expression is $T_{(R,\mu,a,b)}(\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1})$

Link invariant

Theorem (Turaev) *If β and β' present the same link, then* $T_{R,\mu,a,b}(\beta) = T_{R,\mu,a,b}(\beta')$

Precise Definition of the Colored Jones Polynomial

$V := \mathbb{C}^N$

Precise Definition of the Colored Jones Polynomial

$V := \mathbb{C}^N$

$\{e_0, e_1, \ldots, e_{N-1}\}$ is the standard basis of V

Precise Definition of the Colored Jones Polynomial

$V := \mathbb{C}^N$

$\{e_0, e_1, \ldots, e_{N-1}\}$ is the standard basis of V

$R\colon V \otimes V \to V \otimes V$

Precise Definition of the Colored Jones Polynomial

$V := \mathbb{C}^N$

$\{e_0, e_1, \ldots, e_{N-1}\}$ is the standard basis of V

$R\colon V \otimes V \to V \otimes V$

$$R(e_k \otimes e_l) := \sum_{i,j=0}^{N-1} R_{kl}^{ij} e_i \otimes e_j,$$

Precise Definition of the Colored Jones Polynomial

$V := \mathbb{C}^N$

$\{e_0, e_1, \ldots, e_{N-1}\}$ is the standard basis of V

$R\colon V \otimes V \to V \otimes V$

$$R(e_k \otimes e_l) := \sum_{i,j=0}^{N-1} R_{kl}^{ij} e_i \otimes e_j,$$

$$R_{kl}^{ij} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\delta_{k,j-m} \frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \times q^{\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2-m(m+1)/4},$$

Precise Definition of the Colored Jones Polynomial

$V := \mathbb{C}^N$

$\{e_0, e_1, \ldots, e_{N-1}\}$ is the standard basis of V

$R\colon V \otimes V \to V \otimes V$

$$R(e_k \otimes e_l) := \sum_{i,j=0}^{N-1} R_{kl}^{ij} e_i \otimes e_j,$$

$$R_{kl}^{ij} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\delta_{k,j-m} \frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \times q^{\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2-m(m+1)/4},$$

$\{m\} := q^{m/2} - q^{-m/2}$ **and** $\{m\}! := \{1\}\{2\}\cdots\{m\}$

Precise Definition of the Colored Jones Polynomial

$V := \mathbb{C}^N$

$\{e_0, e_1, \ldots, e_{N-1}\}$ is the standard basis of V

$R \colon V \otimes V \to V \otimes V$

$$R(e_k \otimes e_l) := \sum_{i,j=0}^{N-1} R_{kl}^{ij} e_i \otimes e_j,$$

$$R_{kl}^{ij} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\delta_{k,j-m} \frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \times q^{\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2-m(m+1)/4},$$

$\{m\} := q^{m/2} - q^{-m/2}$ **and** $\{m\}! := \{1\}\{2\}\cdots\{m\}$

$\mu \colon V \to V$

Precise Definition of the Colored Jones Polynomial

$V := \mathbb{C}^N$

$\{e_0, e_1, \ldots, e_{N-1}\}$ is the standard basis of V

$R \colon V \otimes V \to V \otimes V$

$$R(e_k \otimes e_l) := \sum_{i,j=0}^{N-1} R_{kl}^{ij} e_i \otimes e_j,$$

$$R_{kl}^{ij} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\delta_{k,j-m} \frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \times q^{\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2-m(m+1)/4},$$

$\{m\} := q^{m/2} - q^{-m/2}$ **and** $\{m\}! := \{1\}\{2\}\cdots\{m\}$

$\mu \colon V \to V$

$$\mu(e_j) := \sum_{i=0}^{N-1} \mu_j^i e_i$$

$$\mu_j^i := \delta_{i,j} q^{(2i-N+1)/2}.$$

Precise Definition of the Colored Jones Polynomial

$(R, \mu, q^{(N^2-1)/4}, 1)$ gives an enhanced Yang–Baxter operator

Precise Definition of the Colored Jones Polynomial

$(R, \mu, q^{(N^2-1)/4}, 1)$ gives an enhanced Yang–Baxter operator

$J_N(L; q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(K) \times \frac{\{1\}}{\{N\}}$: colored Jones polynomial

Example: Figure-eight knot (C. Armond)

$$J_K(N) = q^{(1-N)} \sum_{n=0}^{N-1} \sum_{k=0}^{n} \binom{n}{k}_q q^{n+k(k+1)} [\prod_{j=1}^{n} (1 - q^{j-N})] [\prod_{i=1}^{n-k} (1 - q^{k+i-N})]$$

where $\binom{n}{k}_q$ is the q-binomial coefficient

$$\binom{n}{k}_q = \prod_{i=0}^{k-1} \frac{1 - q^{n-i}}{1 - q^{i+1}}$$

refs

An Introduction to the Volume Conjecture, by Hitoshi Murakami, 2010

Thank You