The Jones Polynomial

Junior Topology Seminar
Mustafa Hajij

Note: Some figures in this file have been taken from the paper : “An Introduction to the Volume Conjecture” by Hitoshi Murakami

Alexander's Theorem

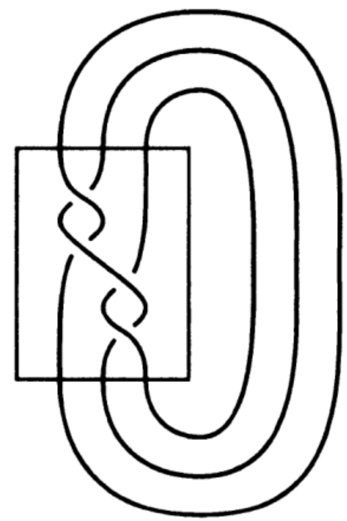
Any tame oriented link L in 3-space may be represented by a pair (b, n), where b is an element of the n-string braid group B_n.

The link *L* is obtained by closing *b*

b

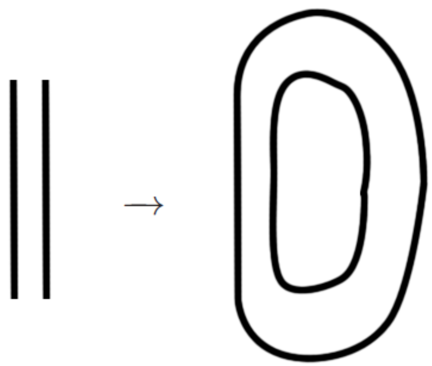
$\longrightarrow$

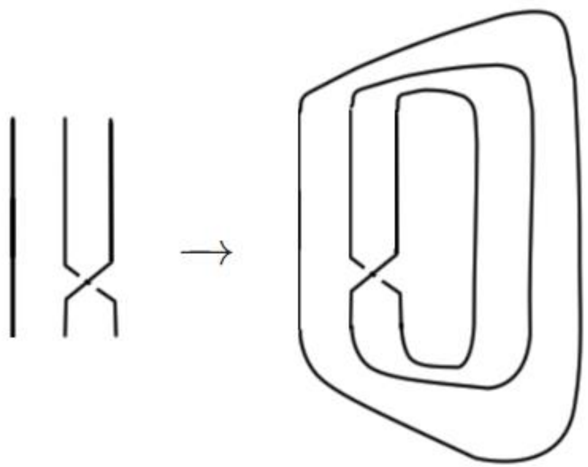
L



Alexander’s Theorem

The problem with Alexander’s theorem is that many differnet braids can give us the same knot.

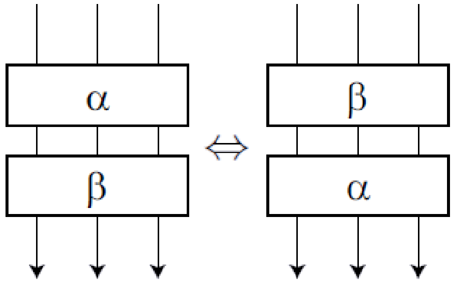




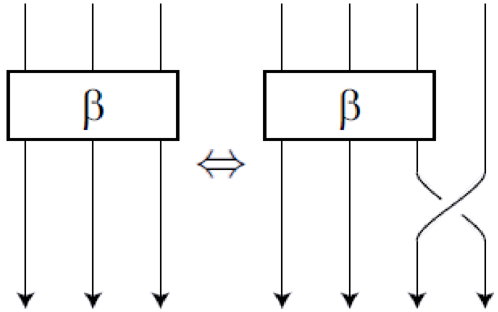
Markov's theorem

b and b' give equivalent links $\iff$ b and b' are related by

conjugation



stabilization



Link invariants

Alexander's and Markov's theorems can be used to define link invariants.

Link invariants

Suppose that we are given a commutative ring R (polynomials or Laurent polynomials for example), and functions $J_n : B_n \to R$ from the n-strand braid group to the ring R, defined for each $n = 2, 3, 4, \ldots$

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(3)If $b \in B_n$, then there is a constant $a \in R$, independent of n, such that

$$\begin{aligned} J_{n+1}(b\sigma_n) &= a^{+1} J_n(b) \\ J_{n+1}(b\sigma_n^{-1}) &= a^{-1} J_n(b) \end{aligned}$$

Link invariants

Let $\{ J_n : B_n \to R\}$ be given with properties 1,2 and 3. For any link L, let $L \backsimeq \hat{b}$, $b \in B$, via Alexander's theorem. If we define $J(L) \in R$ via the formula

$$J(L) = a^{-w(b)} J_n(b)$$

Then $J(L)$ is a link invariant.

Link invariants

Artin's presentation of the braid group is on the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with the relations

$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } 1 \le i \le n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| \ge 2 \end{cases}$$

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Consider the finite-dimensional algebras $A(n,\tau)$ generated by an identity 1 and n projections e_1,e_2,... ,e_{n-1} and satisfy the relations

(1) $e_i^2 = e_i$

(2) $e_i e_{i\pm 1} e_i = \tau e_i$

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Here τ is a complex number. Wenzl showed that such an algebra exists when τ is either real and positive or $e^{\pm\frac{2\pi i}{k}}$ for some $k = 3, 4, 5, \ldots$

Trace operator

There exists a collection of C linear maps $tr_n : A_n \to C$ completely determined by

(1) $tr(1) = 1$
(2)$tr(ab) = tr(ba)$
(3)$tr(we_{n+1}) = \tau tr(w) \quad$ for $w \in A_{n+1}$

A representation for the braid group in A(n)

$$\begin{array}{rcl} B_n & \rightarrow & A_n \\ \sigma_i & \rightarrow & \rho_n(\sigma_i) = a.1_A + be_i \end{array}$$

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where a and b are chosen so that ρ_n is a well defined representation, i.e.,
(1) $a.1_A + be_i$ is a unit in the algebra $A(n,\tau)$
(2)$\rho_n(\sigma_i\sigma^{-1}) = 1$
(2) $\rho_n(\sigma_i\sigma_j) = \rho_n(\sigma j\sigma_i)$
(3) $\rho_n(\sigma_i\sigma_{i+1}\sigma_i) = \rho_n(\sigma i + 1\sigma_i\sigma_{i+1})$

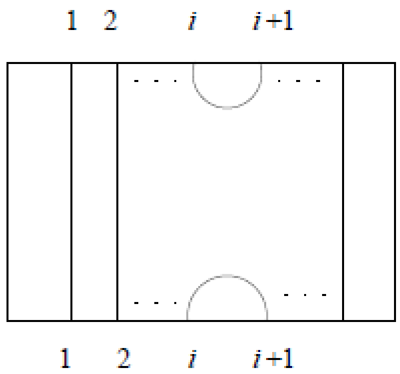
A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS

$$
\begin{array}{ccc}
 & B_n \xrightarrow{\rho_n} A_n \xrightarrow{tr} C \\
\beta & \longrightarrow & \rho_n(\beta) \longrightarrow tr(\rho_n(\beta))
\end{array}
$$

Up to normalization this is a knot invariant. The Jones polynomial.

Diagrammatic realization for A(n)

Consider the algebra, over the complex numbers, that is generared by an identity 1 and the elements E_1,E_2,... ,E_{n-1} where E_i is given by



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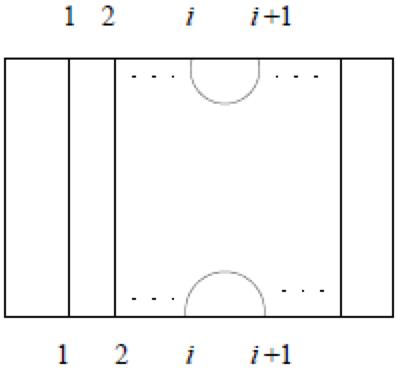

Those generators satisfy the relations

$$E_i^2 = \delta E_i,\ \ E_iE_{i\pm1}E_i = E_i \text{ and } E_iE_j = E_jE_i \text{ for } |i-j| \geq 2.$$

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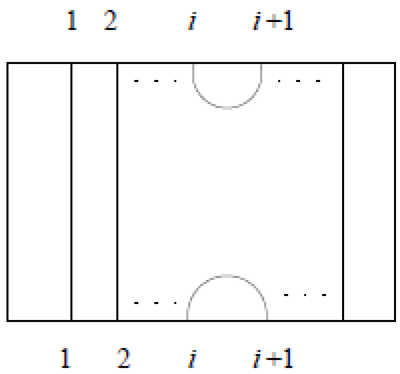

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This algebra is called the temperly-lieb algebra and it is denoted by $Tr(n, \delta)$.

Diagrammatic realization for A(n)

The temperly-lieb algebra $Tr(n, \delta)$ and $A(n, \delta^{-2})$ are isomorphic via the the map that sends e_i to $\frac{1}{\delta} E_i$

A representation for the braid group in the T.L. algebra

Define the map

$$\rho_n : B_n \to Tr(n, \delta)$$

by the formulas:
$\rho_n(\sigma_i) = A(1_n) + A^{-1}E_i$
$\rho_n(\sigma_i^{-1}) = A^{-1}(1_n) + AE_i$

This is also a group representation.

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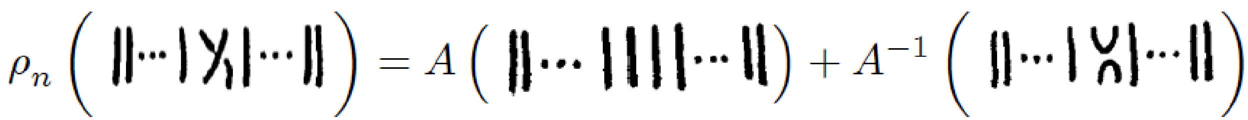

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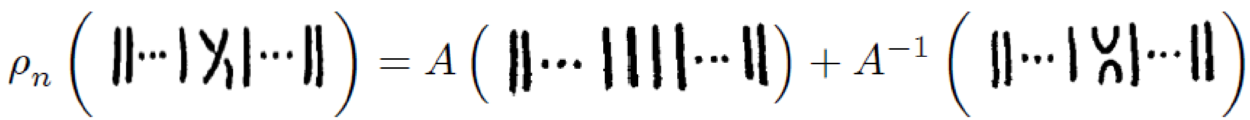
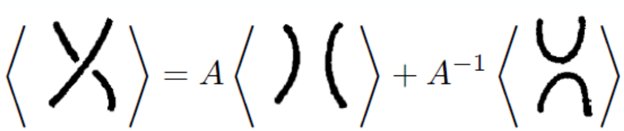


Looks familier?



The trace operator

The trace operator is in this case can be interpreted as the bracket polynomial for $c(\rho_n(\beta))$ the closure of $\rho_n(\beta)$ for some β in B_n. The existence of this trace operator now follows from the existence of the bracker polynomial

So we have the usual Jones-Kauffman polynomial

$$J_K(A) = (-A^3)^{-w(\beta)} < c(\rho_n(\beta)) >$$

Jones polynomial via R matrices

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{t} & 0 \\ 0 & -\sqrt{t} & 1-t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\frac{1}{t} & -\frac{1}{\sqrt{t}} & 0 \\ 0 & -\frac{1}{\sqrt{t}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Define

$$\phi_n : B_n \to M(2^n, \mathbb{Z}[\sqrt{t}, \tfrac{1}{\sqrt{t}}])$$

$$\phi_n(\sigma_i^{\varepsilon}) = I_2 \otimes ... \otimes I_2 \otimes R^{\varepsilon} \otimes I_2 ... I_2$$

The usual trace of this rep is not knot invariant. However this can be fixed

Jones polynomial via R matrices

Let

$$\mu = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}.$$

Denote $\mu \otimes \ldots \otimes \mu$ by $\mu^{\otimes n}$ then $\mu^{\otimes n}$ is a $2^n \times 2^n$

$$tr(\mu^{\otimes n}) = (1+t)^n$$

Jones polynomial via R matrices

$$\phi_n : B_n \to M(2^n, \mathbb{Z}[\sqrt{t}, \frac{1}{\sqrt{t}}])$$

$$\phi_n(\sigma_i^\varepsilon) = I_2 \otimes ... \otimes I_2 \otimes R^\varepsilon \otimes I_2 ... I_2$$

$$J(\beta) = \frac{t^{\frac{1}{2}(\varepsilon(\beta)-n+1)}}{1+t} tr(\phi_n(\beta)\mu^{\otimes n})$$

The latter function is the Jones polynomial

The enhanced Yang-Baxter operator

V: a N-dimensional vector space over $\mathbb{C}$.

$R\colon V \otimes V \to V \otimes V$ (R-matrix), $\mu\colon V \to V$: isomorphisms

a, b: non-zero complex numbers.

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A quadruple (R, μ, a, b) is called an enhanced Yang–Baxter operator if it satisfies the following:

(1) $(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)$,

(2) $R(\mu \otimes \mu) = (\mu \otimes \mu)R$,

(3) $\mathrm{Tr}_2\big(R^{\pm 1}(\mathrm{Id}_V \otimes \mu)\big) = a^{\pm 1} b\,\mathrm{Id}_V$.

The enhanced Yang-Baxter operator

Here $\mathrm{Tr}_k\colon \mathrm{End}(V^{\otimes k}) \to \mathrm{End}(V^{\otimes(k-1)})$ is defined by

$$\mathrm{Tr}_k(f)(e_{i_1}\otimes e_{i_2}\cdots\otimes e_{i_{k-1}}) := \sum_{j_1,j_2,\ldots,j_{k-1},j=0}^{N-1} f^{j_1,j_2,\ldots,j_{k-1},j}_{i_1,i_2,\ldots,i_{k-1},j}(e_{j_1}\otimes e_{j_2}\otimes\cdots\otimes e_{j_{k-1}}\otimes e_j)$$

where $f \in \mathrm{End}(V^{\otimes k})$ is given by

$$f(e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}) = \sum_{j_1,j_2,\ldots,j_k=0}^{N-1} f^{j_1,j_2,\ldots,j_k}_{i_1,i_2,\ldots,i_k}(e_{j_1}\otimes e_{j_2}\otimes\cdots\otimes e_{j_k})$$

and $\{e_0, e_1, \ldots, e_{N-1}\}$ is a basis of V.

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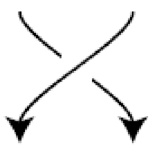
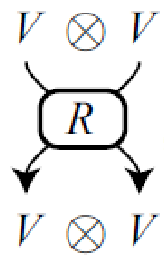
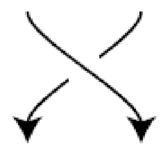
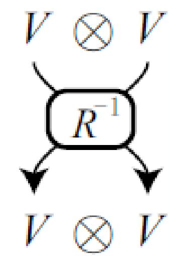
Replace  with 

, and  with 

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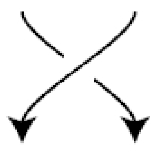
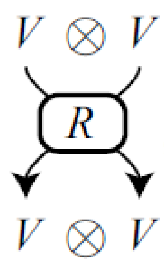
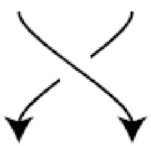
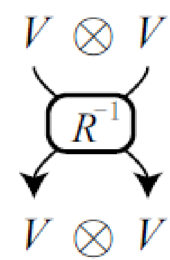
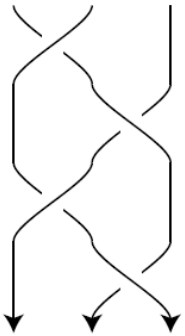
Replace  with 

, and  with 

$$\boxed{n\text{-braid } \beta \Rightarrow \text{homomorphism } \Phi(\beta)\colon V^{\otimes n} \to V^{\otimes n}}$$

Example

$\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$



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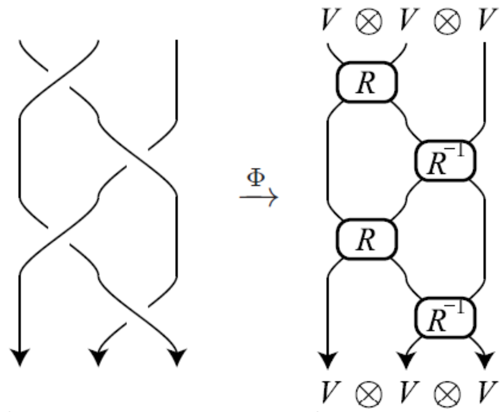

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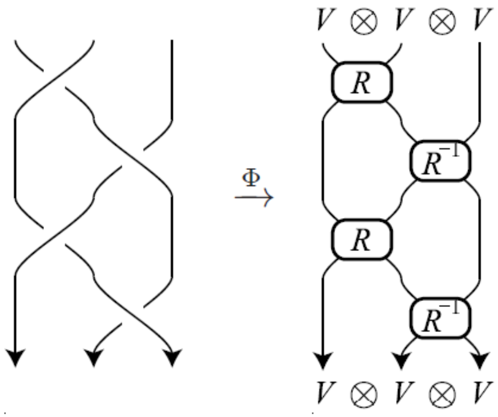

$$\Phi(\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}) = (R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})$$

Trace invariant

n-braid $\beta \Rightarrow$ a link L.

$$T_{(R,\mu,a,b)}(L) := a^{-w(\beta)} b^{-n} \operatorname{Tr}_1\Big(\operatorname{Tr}_2(\cdots(\operatorname{Tr}_n(\Phi(\beta)\mu^{\otimes n}))\cdots)\Big)$$

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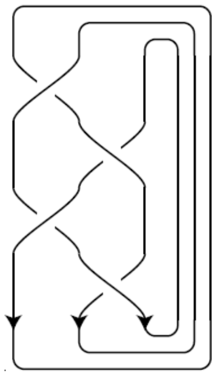

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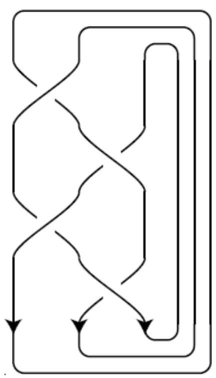
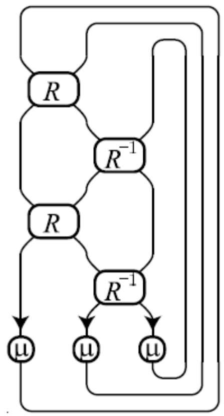


$\Rightarrow$



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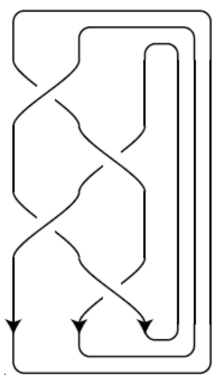

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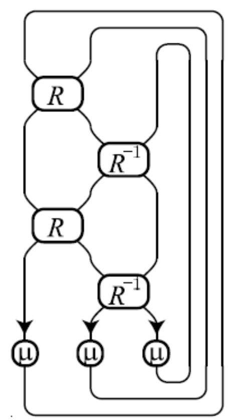

$$\Rightarrow (R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu)$$

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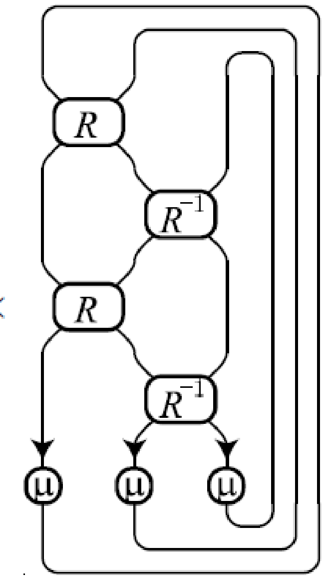
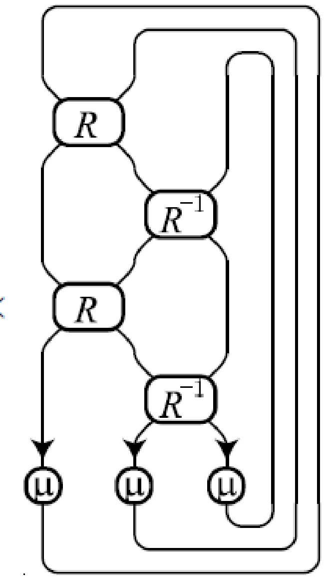


Example

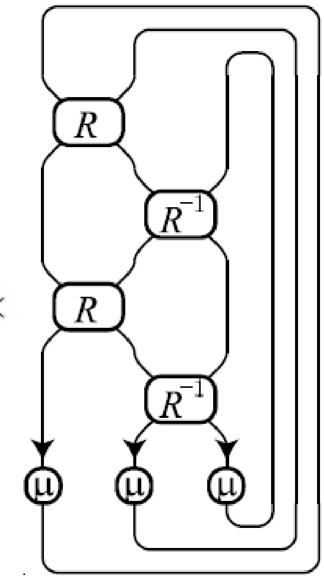
$$(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu) \quad \Rightarrow \quad a^{-w(\beta)} b^{-n} \times$$



$$w(\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}) = +1 - 1 + 1 - 1 = 0$$

Example

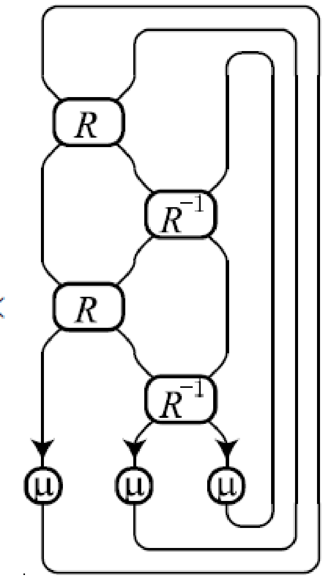
$$(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu) \quad \Rightarrow \quad a^{-w(\beta)} b^{-n} \times$$



$$\Rightarrow \quad b^{-2}\, \mathrm{Tr}_1\big(\mathrm{Tr}_2(\mathrm{Tr}_3((R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu \otimes \mu \otimes \mu)))\big)$$

Example

$$(R\otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R\otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu\otimes\mu\otimes\mu) \quad\Rightarrow\quad a^{-w(\beta)}b^{-n}\times$$



$$\Rightarrow\quad b^{-2}\,\mathrm{Tr}_1\Big(\mathrm{Tr}_2(\mathrm{Tr}_3((R\otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(R\otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R^{-1})(\mu\otimes\mu\otimes\mu)))\Big)$$

And this expression is $T_{(R,\mu,a,b)}(\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1})$

Link invariant

Theorem (Turaev) *If* β *and* β' *present the same link, then* $T_{R,\mu,a,b}(\beta) = T_{R,\mu,a,b}(\beta')$

Refs

An Introduction to the Volume Conjecture, Hitoshi Murakami, *2010.*

Thank You