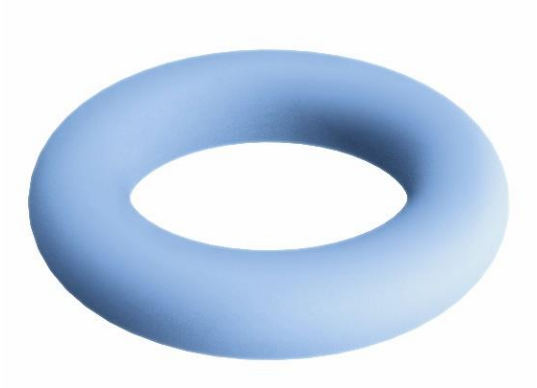
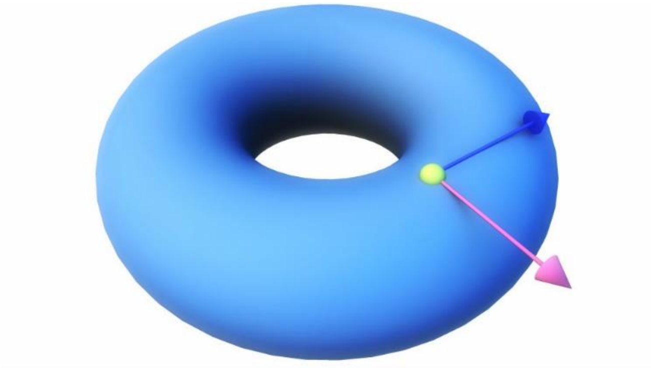
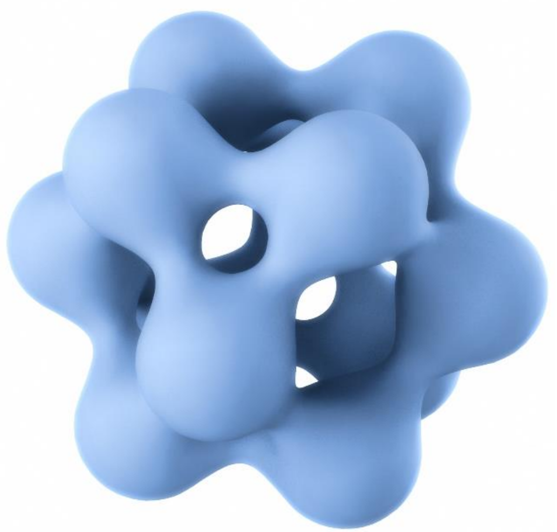


Topology of Surfaces





Mustafa Hajij

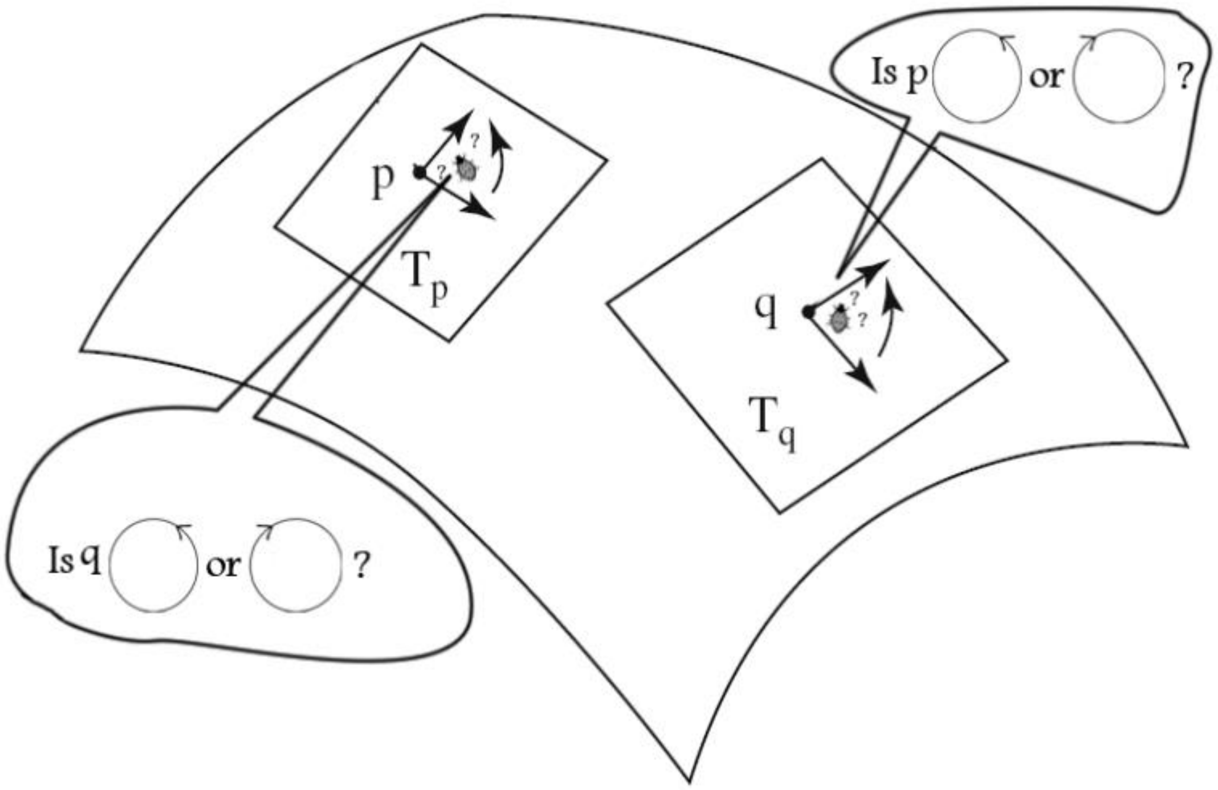
Outline

- Review of surface examples
- Orientable and non-orientable surfaces
- Orientation on meshes
- Connect sum of two surfaces
- Classification of closed surfaces

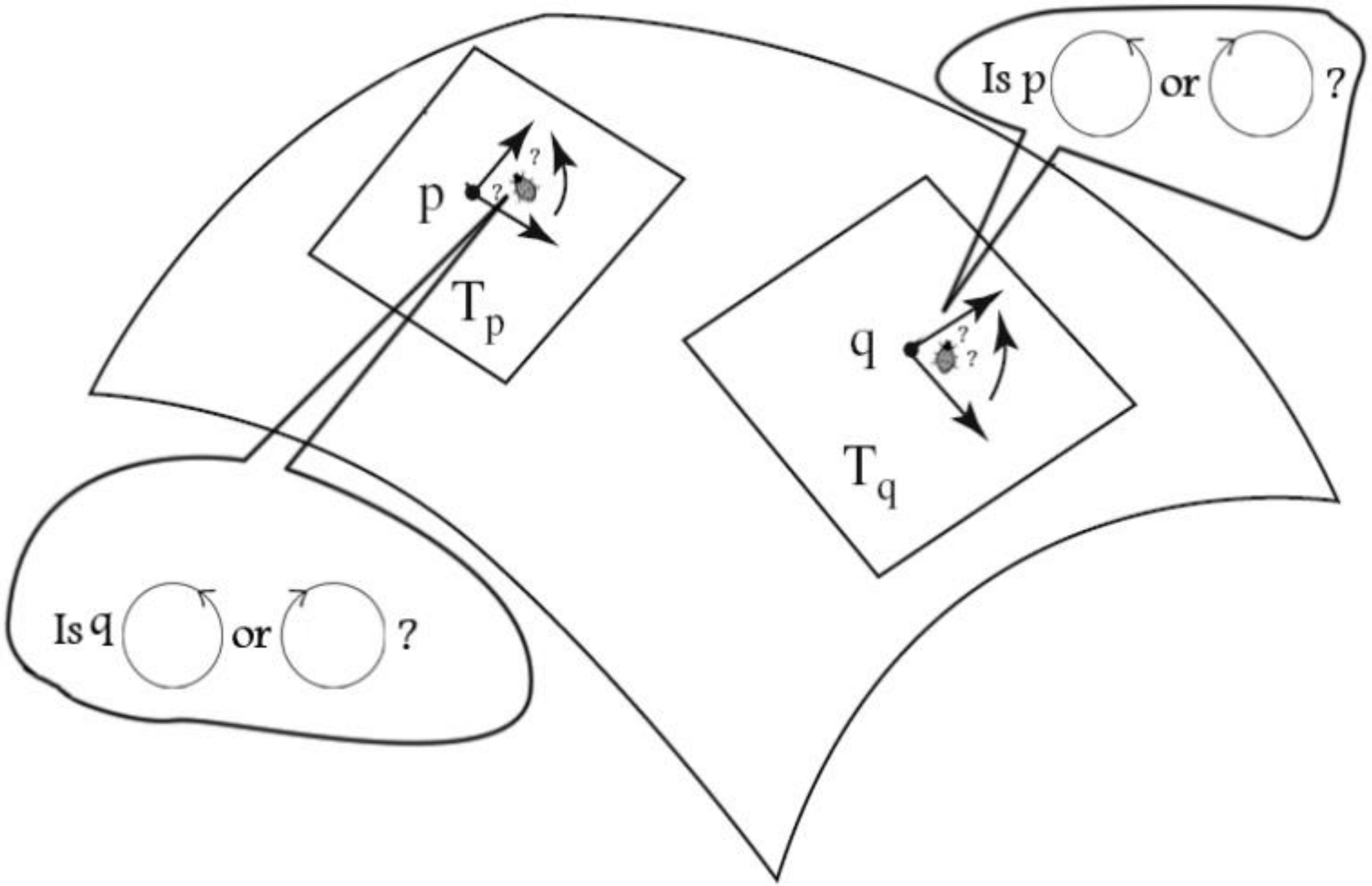
The notion of surface orientation

- An orientable surface is one where it is possible to define a consistent notion of left and right or clockwise and counterclockwise.

- But what does "consistent" mean?



The notion of surface orientation



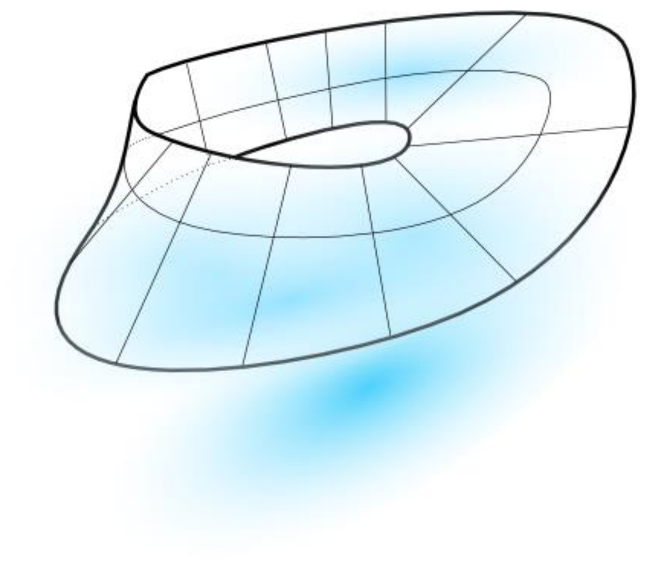
If two persons are standing at different points of a surface and they each have decided what to call clockwise, how can they determine whether their choices are consistent (assuming that they cannot see each other)?

The notion of surface orientation

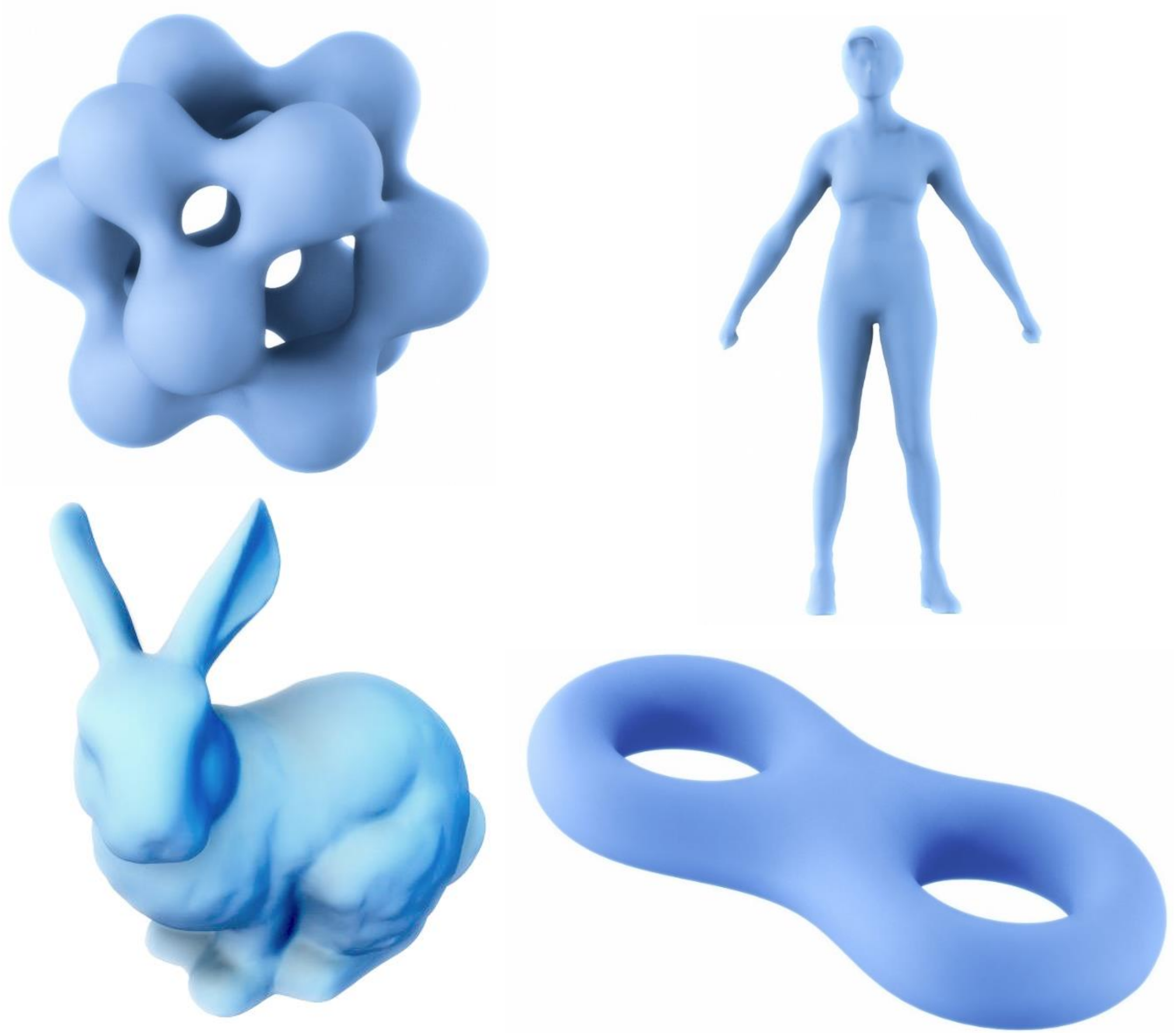
One way to answer this question is to have one of them walk over to where the other one is standing and then compare their notions of clockwise.

The notion of surface orientation

Using this notion, the one can see that the Mobius band is not orientable



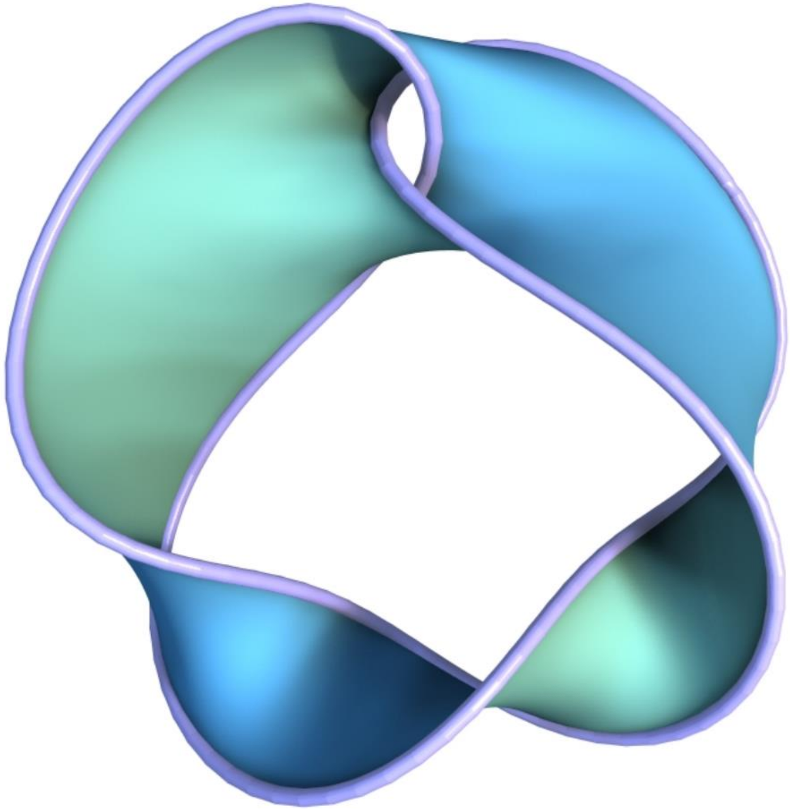
The following surfaces are orientable



Visit mathinsight for a “visual proof” :

http://mathinsight.org/moebius_strip_not_orientable

The notion of surface orientation



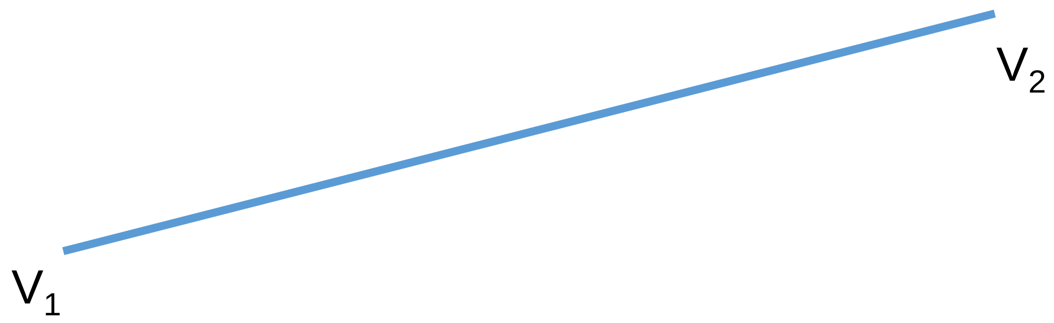
Is this surface orientable?

Orientation on a mesh

How do we specify orientation on a mesh?

To give an answer for this question, we first try to understand the notion of orientation on the simple parts: edges and faces of a mesh.

Given an edge, how can we represent if this edge is oriented from V_1 to V_2 or from V_2 to V_1 ?

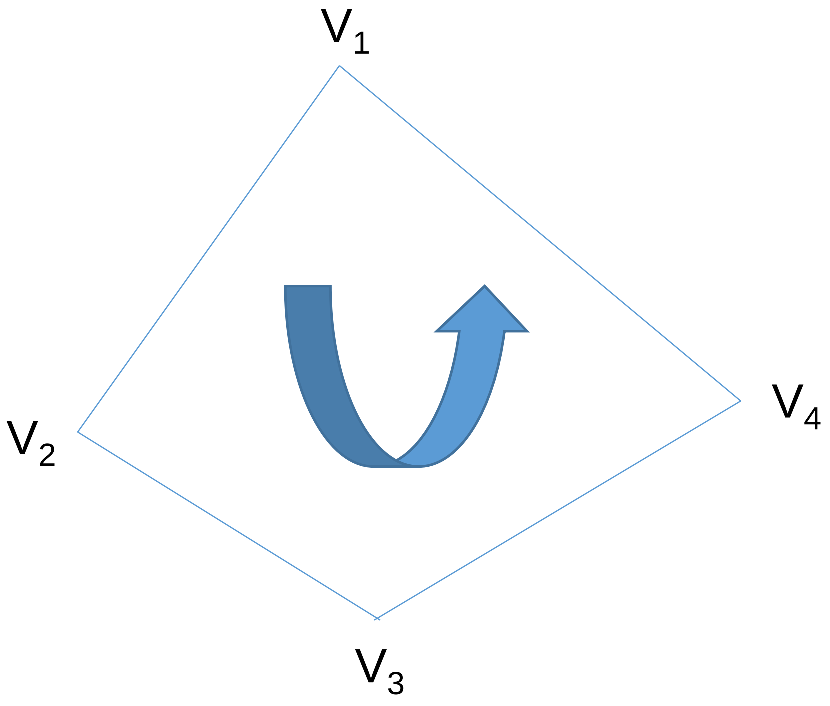


A *choice* of ordering of the vertices can represent orientation of the edge.

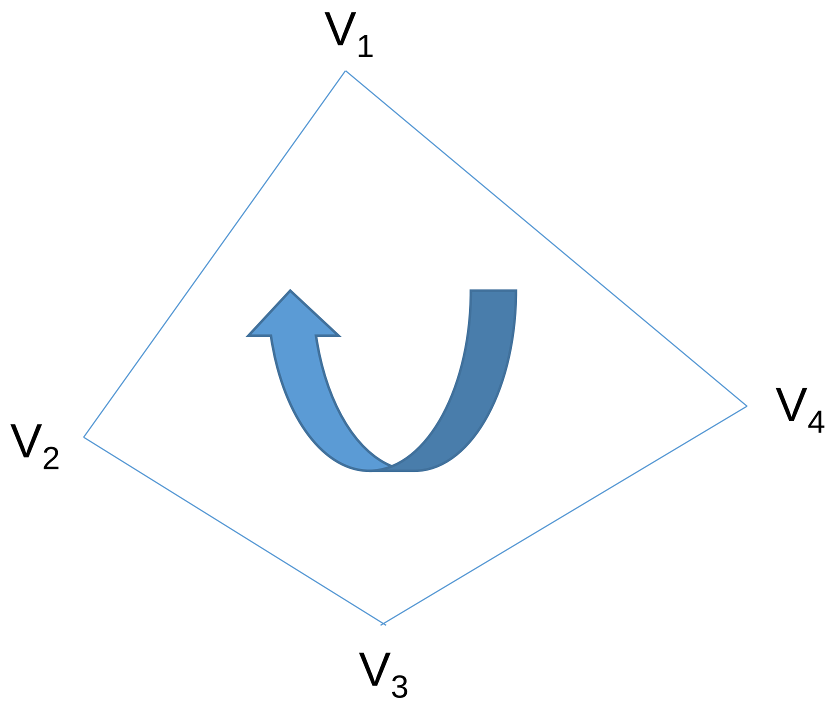
Orientation on a mesh

How do we specify orientation on a mesh?

Given a polygon with vertices $\{V_1,\ldots, V_n\}$, a choice of ordering of the vertices gives an orientation of the polygon.



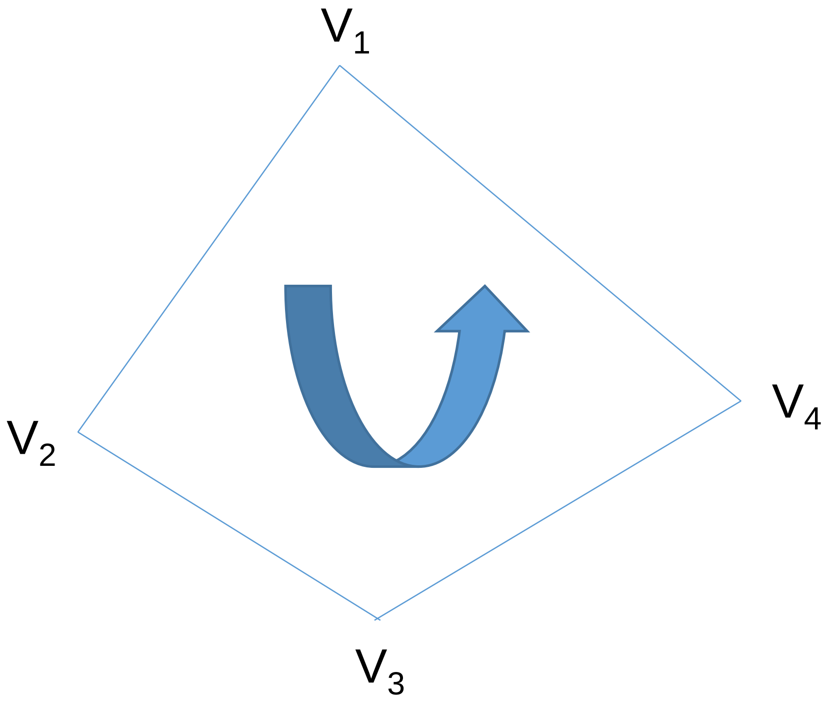
$[V_1, V_2, V_3, V_4]$



$[V_1, V_4, V_3, V_2]$

Orientation on a mesh

How do we specify orientation on a mesh?



Note that the following choices of ordering :

$[V_1, V_2, V_3, V_4]$

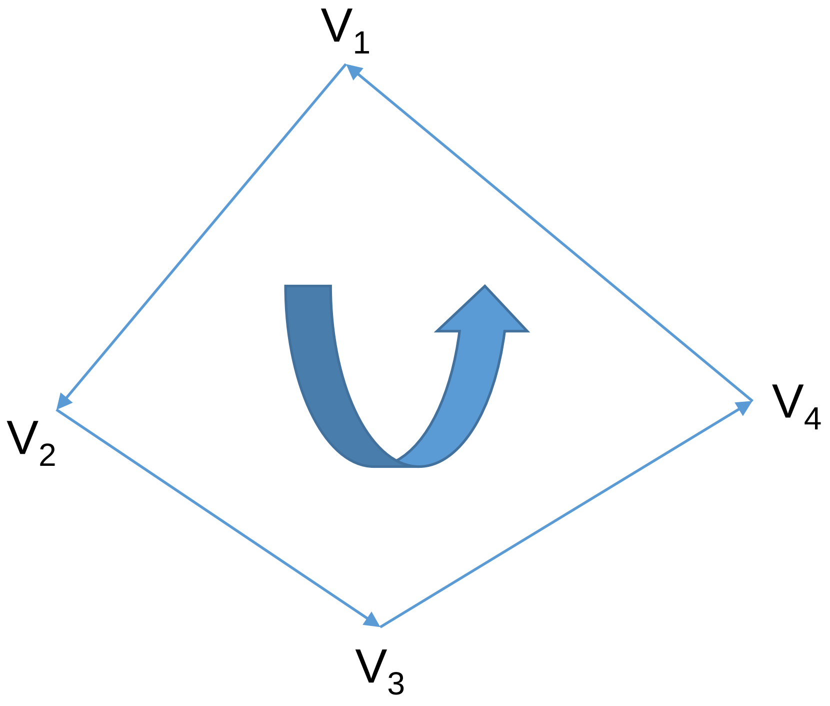
$[V_4, V_1, V_2, V_3]$

Give the same orientation on the polygon $\{V_1, V_2, V_3, V_4\}$

Orientation on a mesh

How do we specify orientation on a mesh?

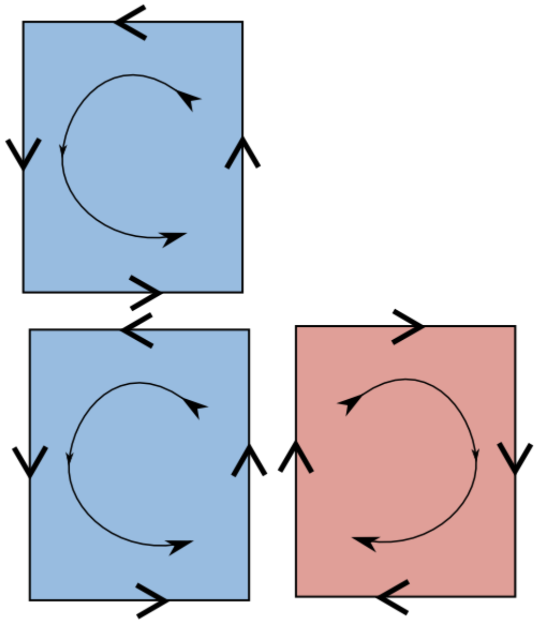
An orientation on the polygon $\{V_1, V_2, V_3, V_4\}$ induces unique orientation on the edges of the polygon



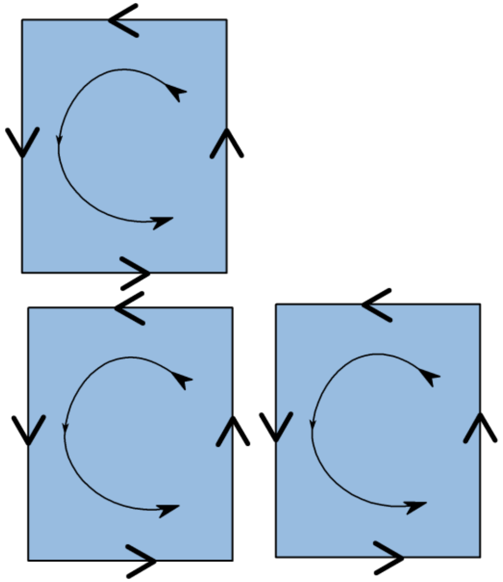
Orientation on a mesh

How do we specify orientation on a mesh?

Now that we understand how to make an orientable face, how can we glue two oriented faces so that the result face is also orientable :



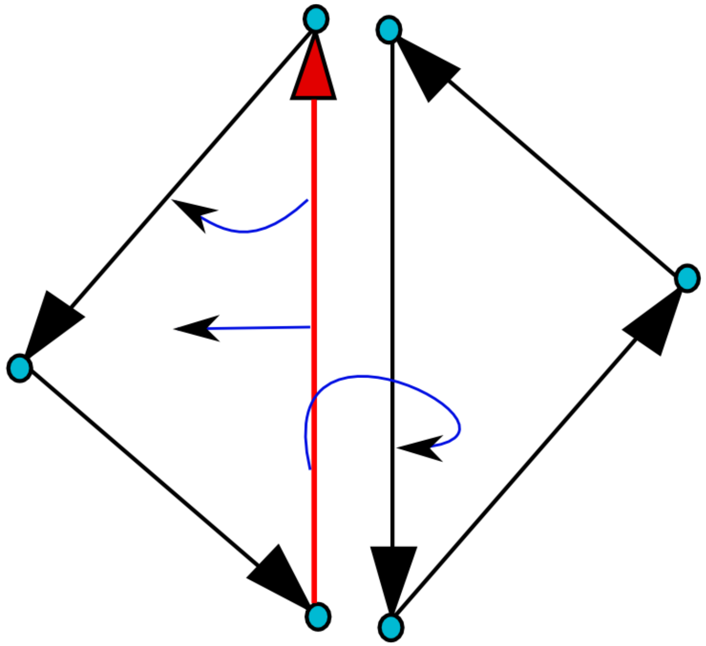
Cannot be glued together because orientations in adjacent faces are inconsistent



Can be glued together to form an oriented mesh

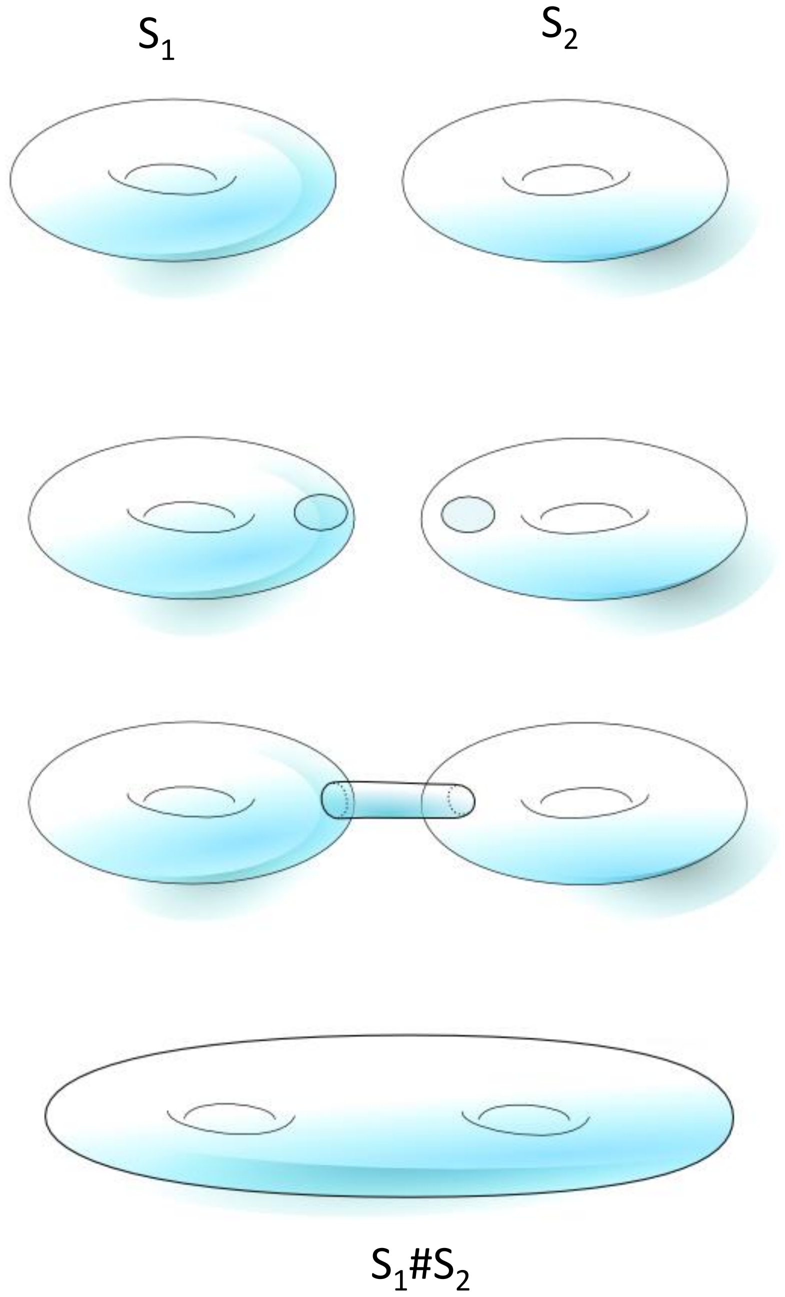
Orientation on a mesh

We conclude that : with the way we defined the halfedge data structure can only represent orientable surfaces



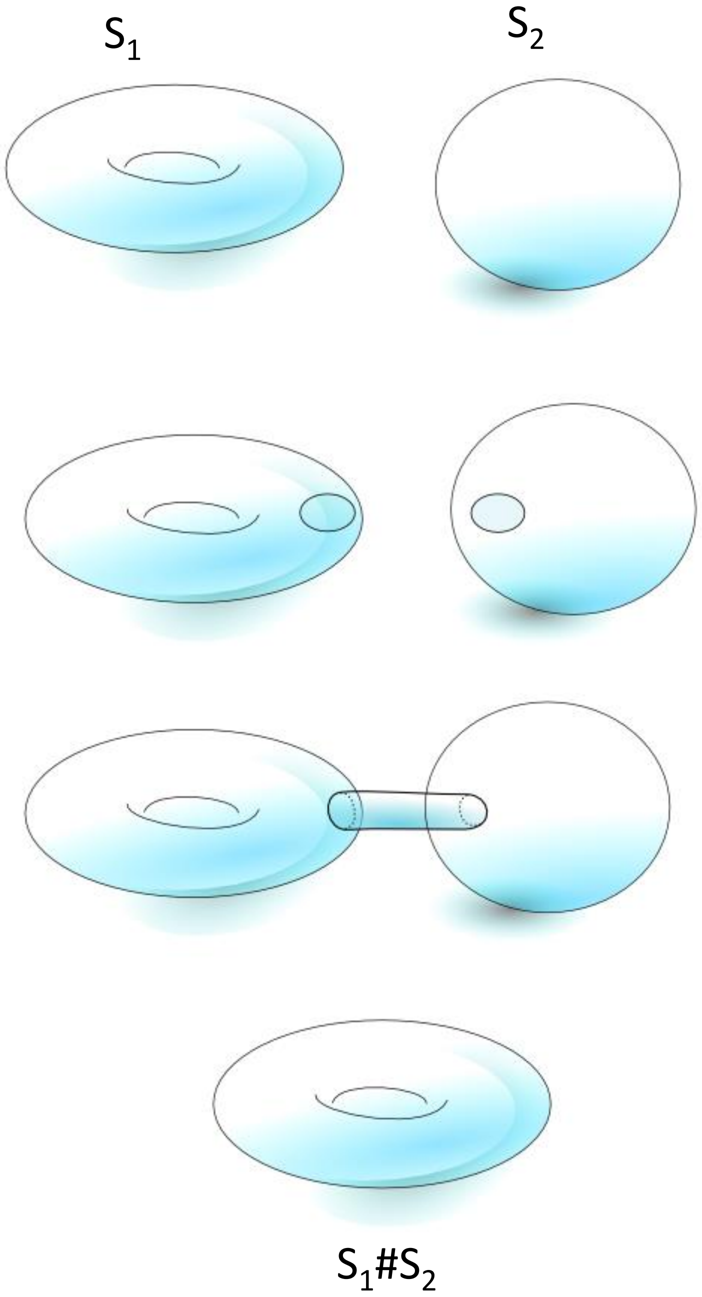
Connect sum of two surfaces

We choose small discs from two connected surfaces S_1, S_2 then and glue the boundary circles together to a form a new surface $S_1\#S_2$



Connect sum of two surfaces

Note that if one of the surfaces S_1, $S_{2,}$ say S_1 is a sphere, is a sphere then then $S_1\#S_2$ is equivalent topologically to $S_{2.}$

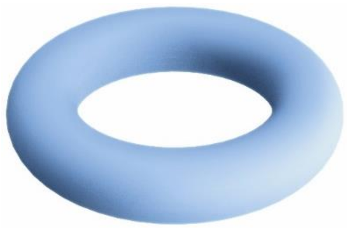


Classification theorem of surfaces

Any compact, connected, orientable and without boundary surface is topologically equivalent to one of the following surfaces :

1. The sphere.

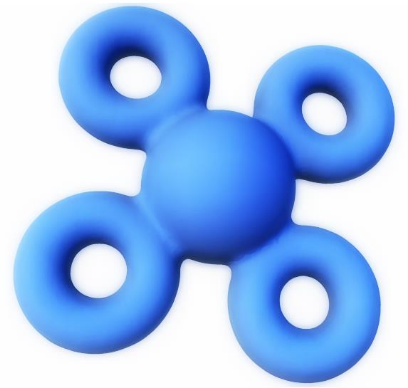
2. The connected sum of g tori, for g ≥ 1.



The connected sum of 1 tori

The connected sum of 2 tori

The connected sum of 3 tori



The connected sum of 4 tori

Classification theorem of surfaces with boundary

Any compact, connected, orientable and with boundary surface is topologically equivalent to one of the following surfaces :

1. The sphere with possibly finite number of disks removed.

2. The connected sum of g tori, for $g \geq 1$, with possibly finite number of disks removed.

In other words, any compact orientable surface with or without a boundary is classified according to:

1- The number of its boundary components
2- Its genus