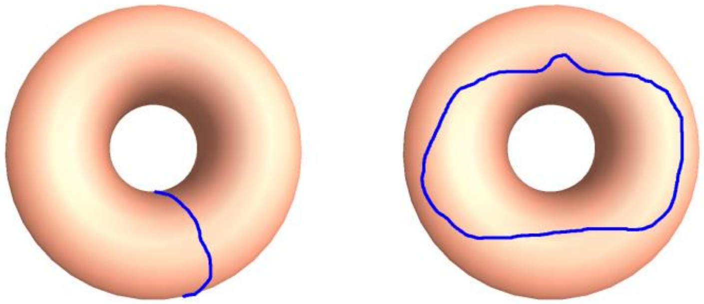
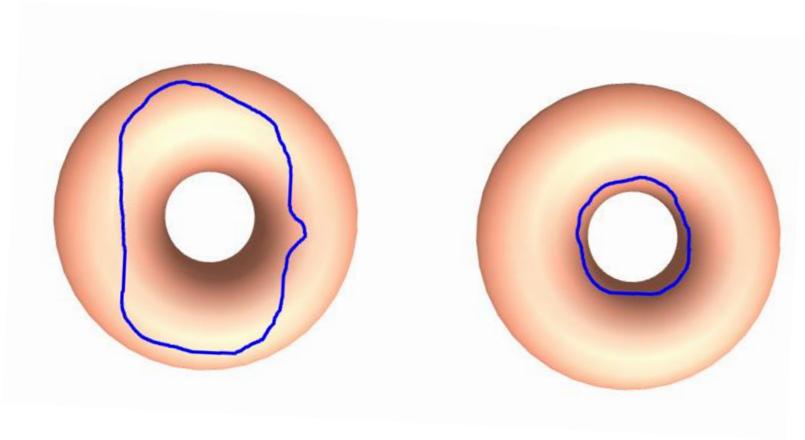
Topological Algorithms-III

Mustafa Hajij

Homotopy detection



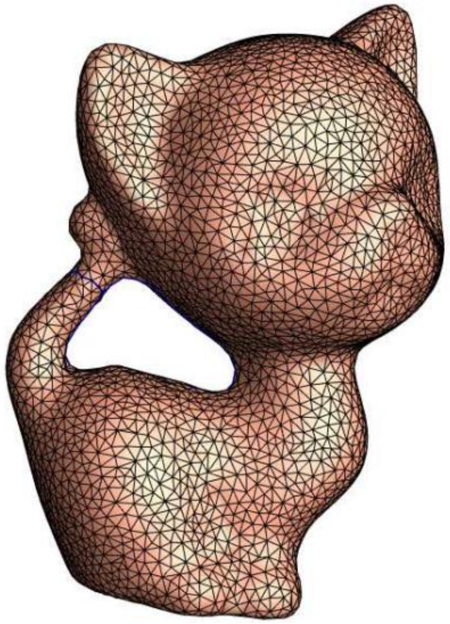
Non-homotopic curves



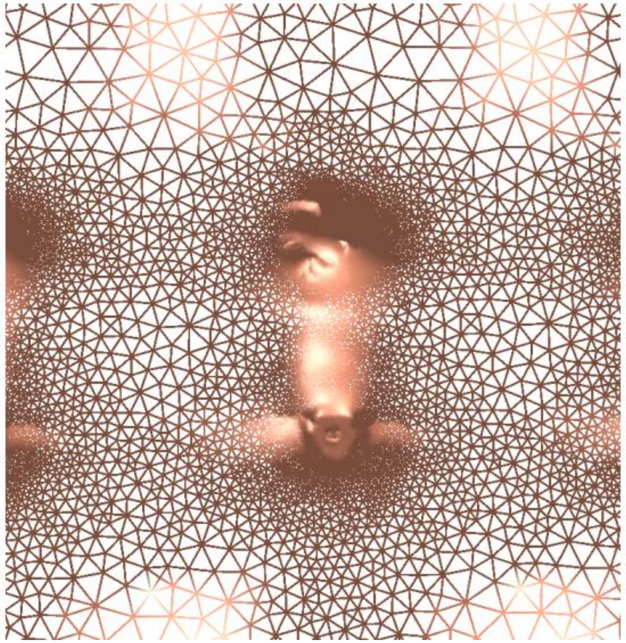
Homotopic curves

Image :David Gu

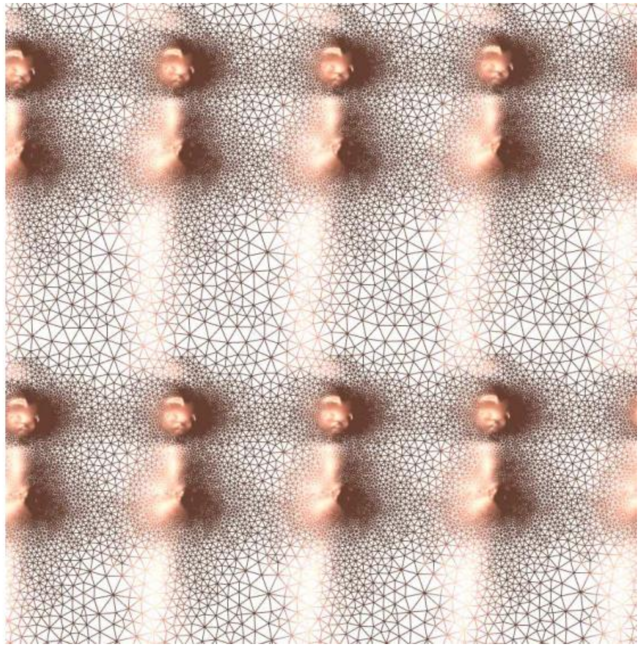
Constructing the universal cover



Input mesh



Fundamental domain



Finite portion of the universal cover

Image :David Gu

Curve lifting

A curve in the original surface can be lifted to the universal covering space

Curve lifting

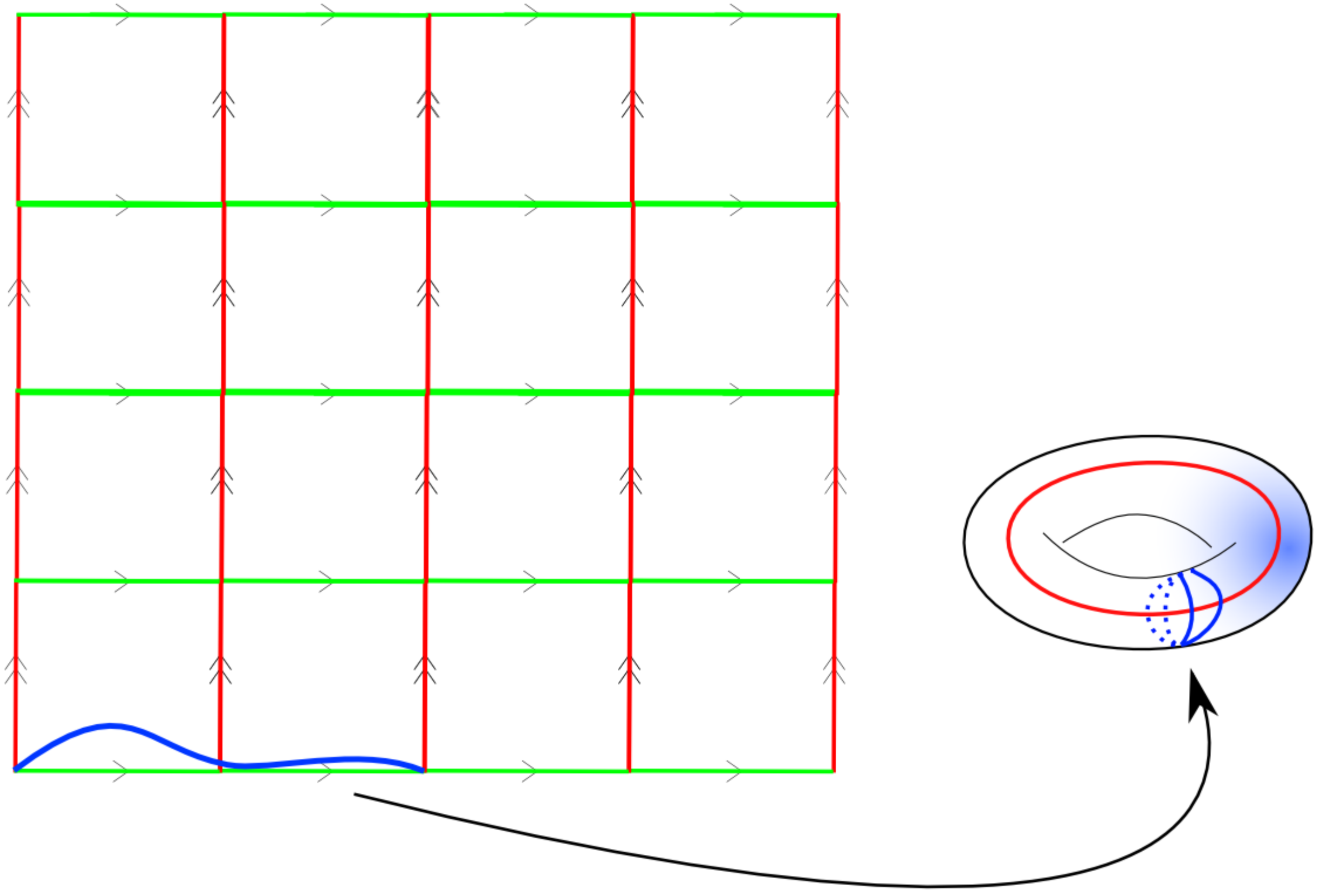
A curve in the original surface can be lifted to the universal covering space

A closed non-trivial loop can be lifted to an open path in the universal covering space

Curve lifting

A curve in the original surface can be lifted to the universal covering space

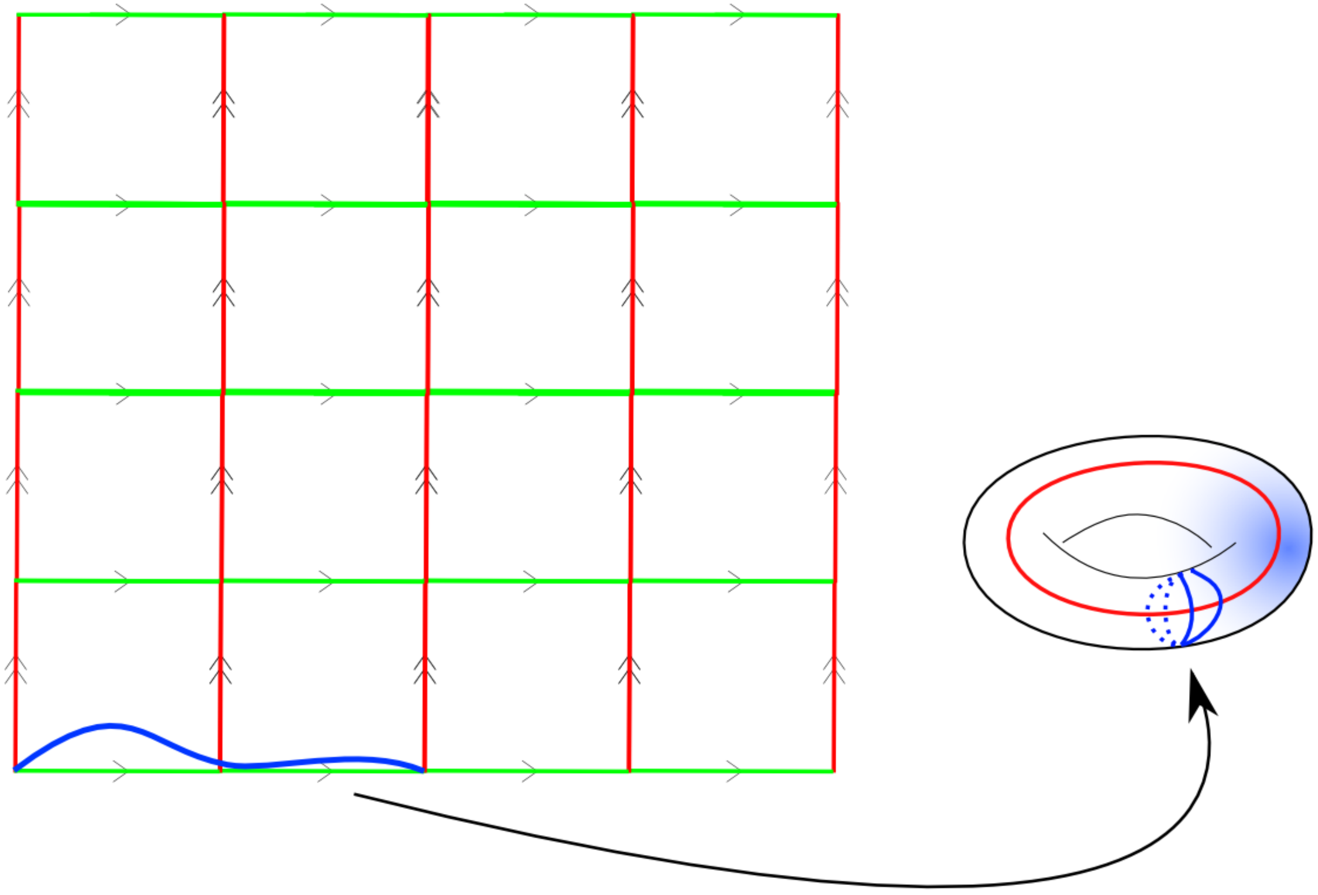
A closed non-trivial loop can be lifted to an open path in the universal covering space



Curve lifting

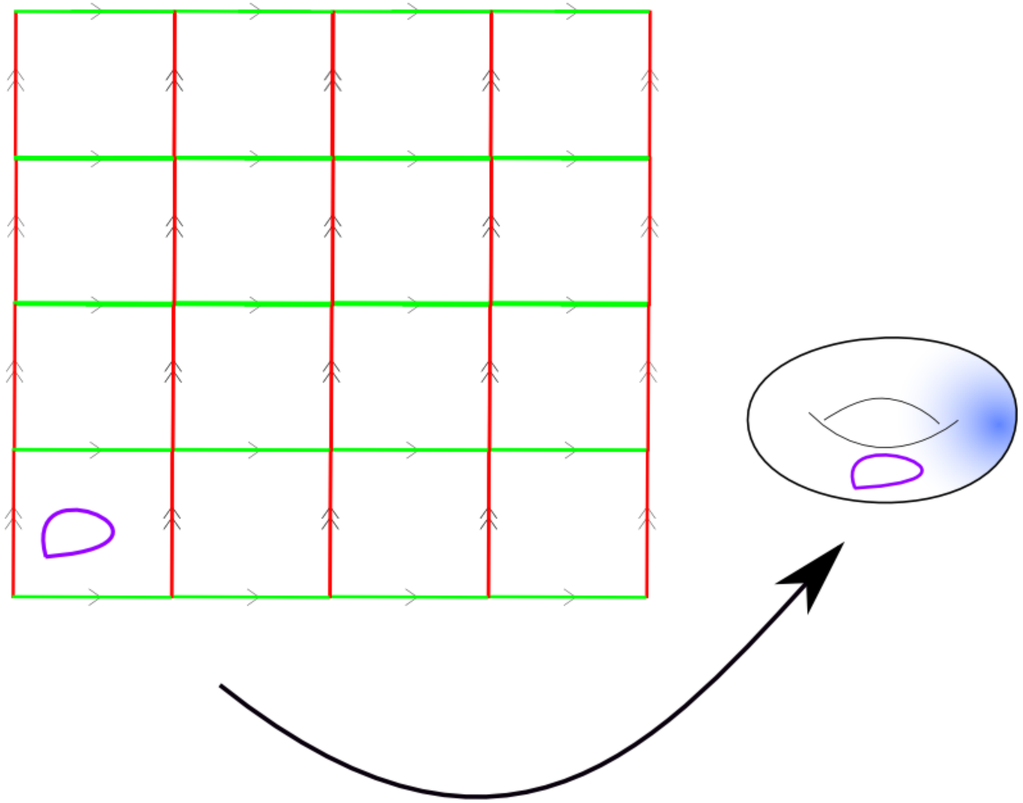
A curve in the original surface can be lifted to the universal covering space

A closed non-trivial loop can be lifted to an open path in the universal covering space



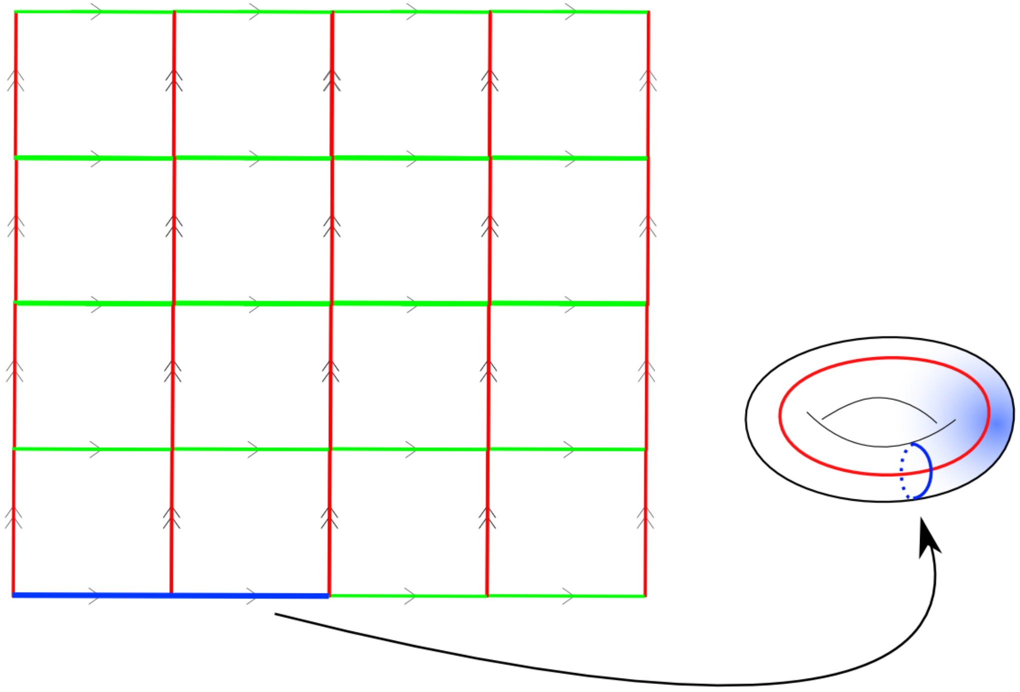
Key idea : many topological problems can be solved on the universal cover easier than on the original surface.

Homotopy detection



Homotopically trivial loops are lifted to closed loops in the covering space.

Homotopy detection



Homotopically non-trivial loops are lifted to open curves in the covering space.

The Shortest Loop

Given a non-trivial *a* loop on a surface, write an algorithm that computes a loop *b* in the same homotopy class of *a* such that *b* is as short as possible.

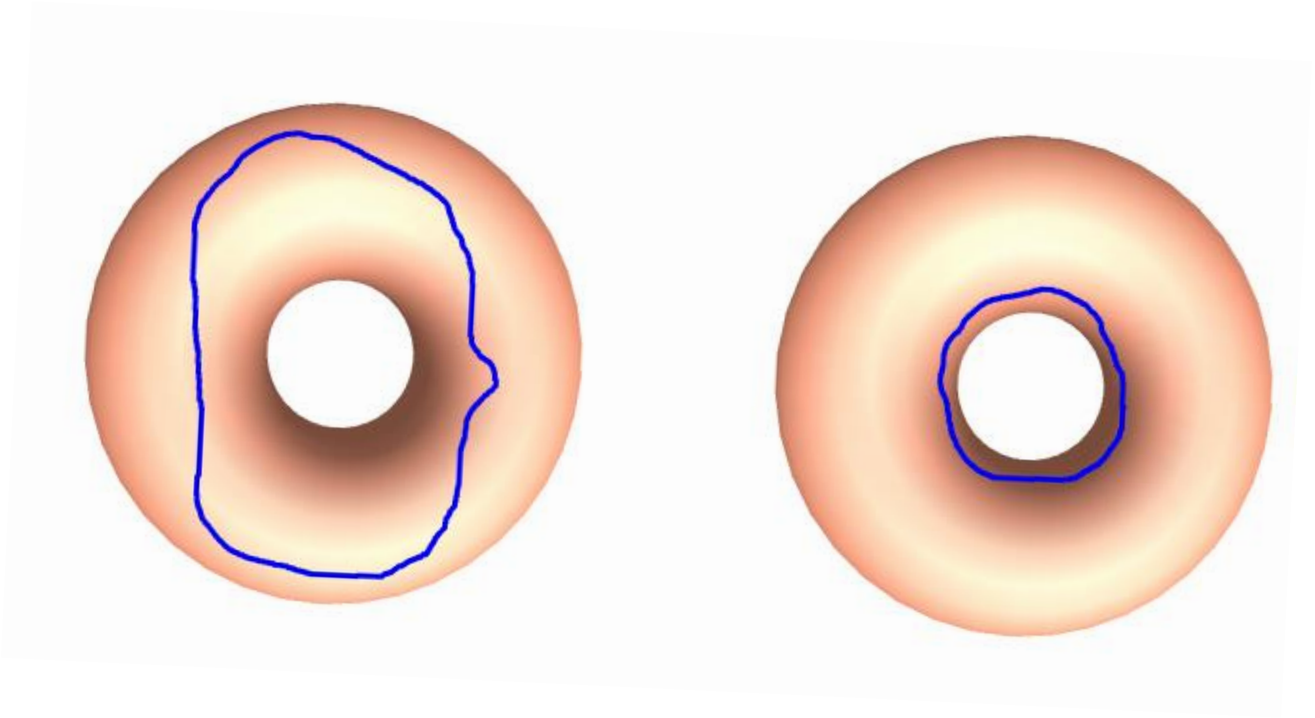
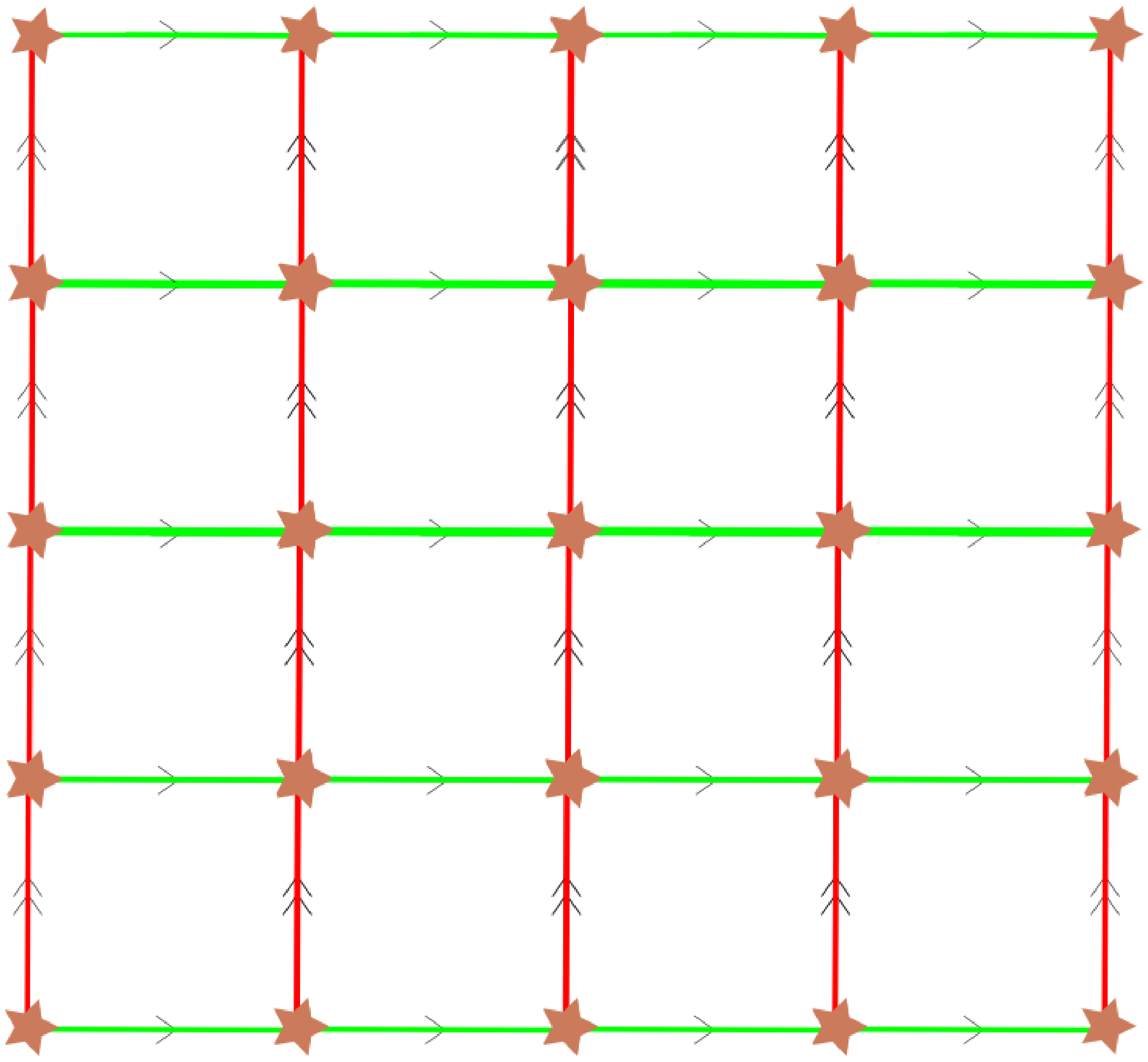


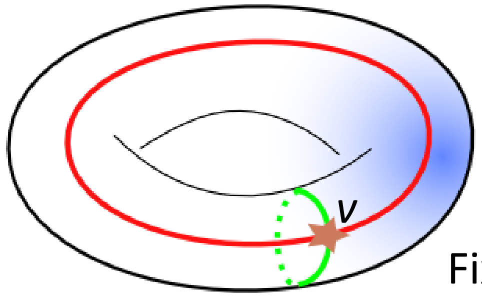
Image :David Gu

The Shortest Loop

Given a non-trivial *a* loop on a surface, write an algorithm that computes a loop *b* in the same homotopy class of *a* such that *b* is as short as possible.

The universal cover and the fundamental domain





Fix a vertex *v* and loop *a* that passes through and we want to find the shortest loop homotopic to *a* that passes through *v*.

Key Idea

In general let $\bar{M}$ be the universal covering space of of a surface M and let $p : \bar{M} \to M$ be a covering map.

Key Idea

In general let $\bar{M}$ be the universal covering space of of a surface M and let $p : \bar{M} \to M$ be a covering map. Then there is a one to one correspondence between the pre-image of q, $p^{-1}(q)$, and the the equivalenc homotopy classes of loops in M starting at $q, \pi(M, q)$.

Key Idea

In general let $\bar{M}$ be the universal covering space of of a surface M and let $p : \bar{M} \to M$ be a covering map. Then there is a one to one correspondence between the pre-image of q, $p^{-1}(q)$, and the the equivalenc homotopy classes of loops in M starting at $q, \pi(M, q)$.

$$\phi : p^{-1}(q) \to \pi(M, q)$$

Key Idea

In general let $\bar{M}$ be the universal covering space of of a surface M and let $p : \bar{M} \to M$ be a covering map. Then there is a one to one correspondence between the pre-image of q, $p^{-1}(q)$, and the the equivalenc homotopy classes of loops in M starting at $q, \pi(M, q)$.

$$\phi : p^{-1}(q) \to \pi(M, q)$$

Fix a point $\hat{q}_0$ in $p^{-1}(q)$ for any $\hat{q}_k$ in $p^{-1}(q)$ we can find a path $\hat{\gamma} : I \to \bar{M}$ connecting $\hat{q}_0$ and $\hat{q}_k$.

Key Idea

In general let $\bar{M}$ be the universal covering space of of a surface M and let $p : \bar{M} \to M$ be a covering map. Then there is a one to one correspondence between the pre-image of q, $p^{-1}(q)$, and the the equivalenc homotopy classes of loops in M starting at $q, \pi(M, q)$.

$$\phi : p^{-1}(q) \to \pi(M, q)$$

Fix a point $\hat{q}_0$ in $p^{-1}(q)$ for any $\hat{q}_k$ in $p^{-1}(q)$ we can find a path $\hat{\gamma} : I \to \bar{M}$ connecting $\hat{q}_0$ and $\hat{q}_k$. The projection of $\hat{\gamma}$ is a loop in M that passes from q.

$$\phi(\hat{q}_k) = [p(\hat{\gamma})]$$

Idea of the algorithm

Assume that $\bar{M}$ is the universal cover of M and $p : \bar{M} \to M$ is the projection.

Idea of the algorithm

Assume that $\bar{M}$ is the universal cover of M and $p : \bar{M} \to M$ is the projection. The preimage of the vertex v is the set

$$p^{-1}(q) = \{\bar{v}_0, \bar{v}_1, ...\}$$

Idea of the algorithm

Assume that $\bar{M}$ is the universal cover of M and $p : \bar{M} \to M$ is the projection. The preimage of the vertex v is the set

$$p^{-1}(q) = \{\bar{v}_0, \bar{v}_1, ...\}$$

Suppose that γ is a non-trivial loops that passes through v. Then we left γ to the universal cover space of M, $\bar{\gamma} \subset \bar{M}$ such that

$$\partial\bar{\gamma} = \{\bar{v}_0, \bar{v}_k\}$$

Idea of the algorithm

Assume that $\bar{M}$ is the universal cover of M and $p : \bar{M} \to M$ is the projection. The preimage of the vertex v is the set

$$p^{-1}(q) = \{\bar{v}_0, \bar{v}_1, ...\}$$

Suppose that γ is a non-trivial loops that passes through v. Then we left γ to the universal cover space of M, $\bar{\gamma} \subset \bar{M}$ such that

$$\partial\bar{\gamma} = \{\bar{v}_0, \bar{v}_k\}$$

Any other loop γ_1 that passes through the vertex v and homotopic to γ will also satisfy

$$\partial\bar{\gamma}_1 = \{\bar{v}_0, \bar{v}_k\}$$

Idea of the algorithm

Assume that $\bar{M}$ is the universal cover of M and $p : \bar{M} \to M$ is the projection. The preimage of the vertex v is the set

$$p^{-1}(q) = \{\bar{v}_0, \bar{v}_1, ...\}$$

Suppose that γ is a non-trivial loops that passes through v. Then we left γ to the universal cover space of M, $\bar{\gamma} \subset \bar{M}$ such that

$$\partial\bar{\gamma} = \{\bar{v}_0, \bar{v}_k\}$$

Any other loop γ_1 that passes through the vertex v and homotopic to γ will also satisfy

$$\partial\bar{\gamma}_1 = \{\bar{v}_0, \bar{v}_k\}$$

And vice versa, any path in the universal cover that connects between $\bar{v}_0$ and $\bar{v}_k$ lifts to a loop at v that is homotopic to γ.

Idea of the algorithm

Assume that $\bar{M}$ is the universal cover of M and $p : \bar{M} \to M$ is the projection. The preimage of the vertex v is the set

$$p^{-1}(q) = \{\bar{v}_0, \bar{v}_1, ...\}$$

Suppose that γ is a non-trivial loops that passes through v. Then we left γ to the universal cover space of M, $\bar{\gamma} \subset \bar{M}$ such that

$$\partial\bar{\gamma} = \{\bar{v}_0, \bar{v}_k\}$$

Any other loop γ_1 that passes through the vertex v and homotopic to γ will also satisfy

$$\partial\bar{\gamma}_1 = \{\bar{v}_0, \bar{v}_k\}$$

And vice versa, any path in the universal cover that connects between $\bar{v}_0$ and $\bar{v}_k$ lifts to a loop at v that is homotopic to γ.

The shortest loop though v corresponds to the shortest path connecting $\bar{v}_0$ and $\bar{v}_k$.

The shortest loop through a given point

Input : A base vertex v a non-trivial loop γ through v on a mesh M

Output: the shortest loop through v and homotopic to γ

The shortest loop through a given point

Input : A base vertex v a non-trivial loop γ through v on a mesh M
Output: the shortest loop through v and homotopic to γ
1-Compute a finite portion of the universal cover $\bar{M}$ of M.

The shortest loop through a given point

Input : A base vertex v a non-trivial loop γ through v on a mesh M

Output: the shortest loop through v and homotopic to γ

1-Compute a finite portion of the universal cover $\bar{M}$ of M.

2- Compute the preimage of v

$$p^{-1}(v) = \{\bar{v}_0, \bar{v}_1, ...\}$$

The shortest loop through a given point

Input : A base vertex v a non-trivial loop γ through v on a mesh M
Output: the shortest loop through v and homotopic to γ
1-Compute a finite portion of the universal cover $\bar{M}$ of M.
2- Compute the preimage of v

$$p^{-1}(v) = \{\bar{v}_0, \bar{v}_1, ...\}$$

3-Lift γ to $\bar{\gamma}$ in $\bar{M}$ such that

$$\partial\bar{\gamma} = \{\bar{v}_0, \bar{v}_k\}$$

The shortest loop through a given point

Input : A base vertex v a non-trivial loop γ through v on a mesh M
Output: the shortest loop through v and homotopic to γ
1-Compute a finite portion of the universal cover $\bar{M}$ of M.
2- Compute the preimage of v

$$p^{-1}(v) = \{\bar{v}_0, \bar{v}_1, ...\}$$

3-Lift γ to $\bar{\gamma}$ in $\bar{M}$ such that

$$\partial\bar{\gamma} = \{\bar{v}_0, \bar{v}_k\}$$

4-Compute the shortest path $\bar{\Gamma}$ in the universal cover space $\bar{M}$

$$\bar{\Gamma} = \{\bar{v}_0, \bar{v}_k\}$$

using Dijkstra's algorithm.

The shortest loop through a given point

Input : A base vertex v a non-trivial loop γ through v on a mesh M
Output: the shortest loop through v and homotopic to γ
1-Compute a finite portion of the universal cover $\bar{M}$ of M.
2- Compute the preimage of v

$$p^{-1}(v) = \{\bar{v}_0, \bar{v}_1, ...\}$$

3-Lift γ to $\bar{\gamma}$ in $\bar{M}$ such that

$$\partial\bar{\gamma} = \{\bar{v}_0, \bar{v}_k\}$$

4-Compute the shortest path $\bar{\Gamma}$ in the universal cover space $\bar{M}$

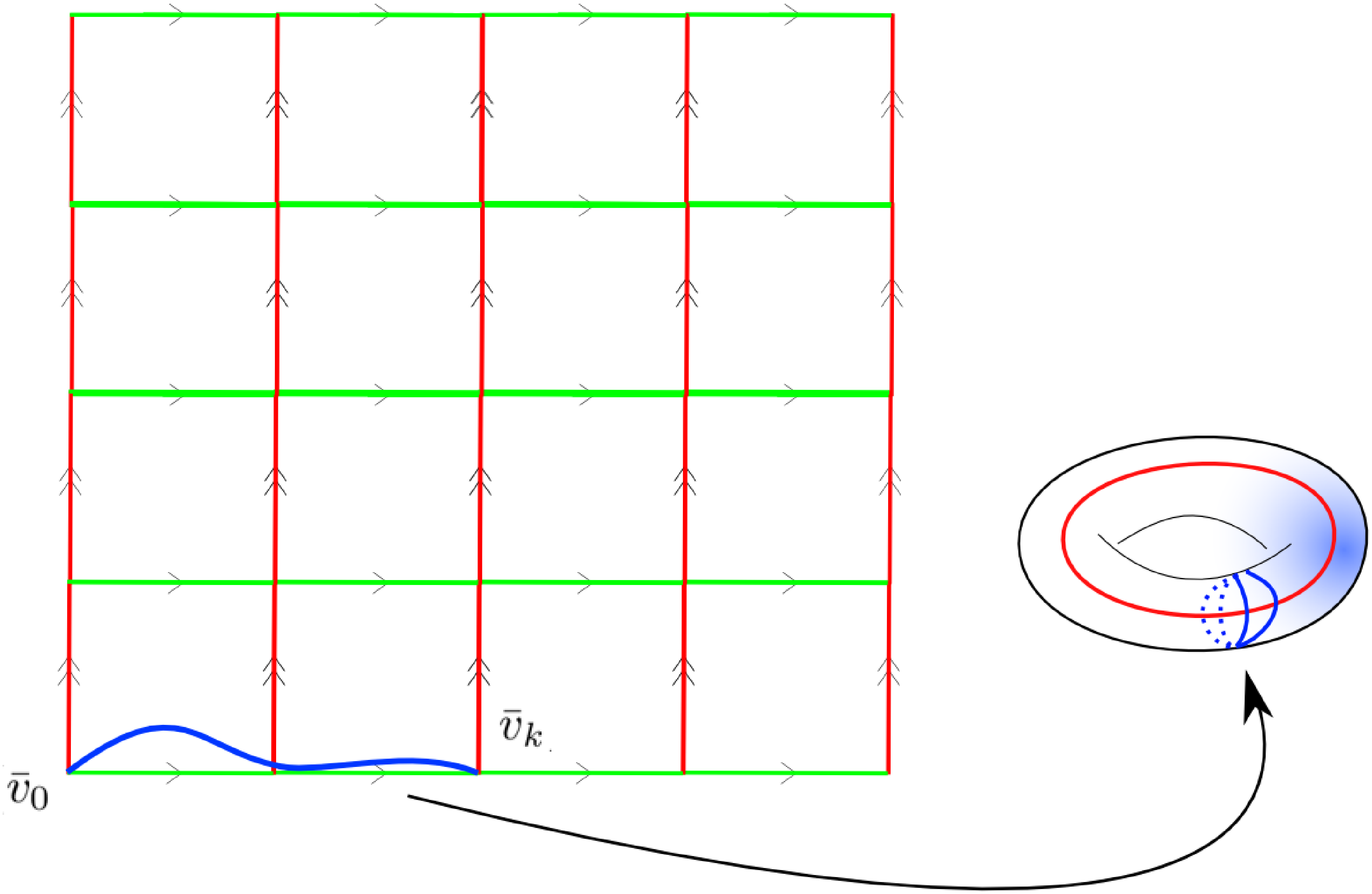
$$\bar{\Gamma} = \{\bar{v}_0, \bar{v}_k\}$$

using Dijkstra's algorithm.
5-The projection of the shortest path

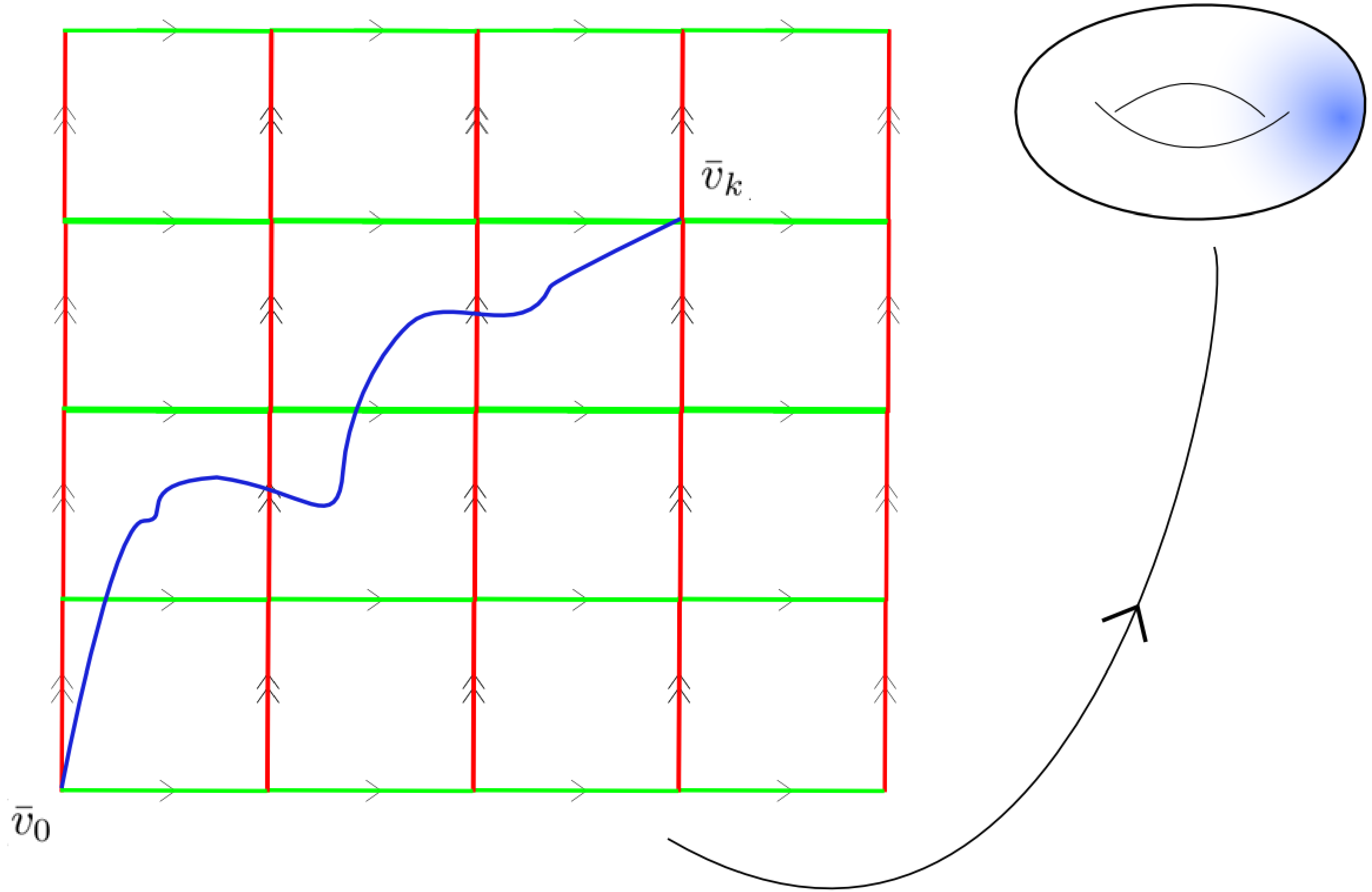
$$\Gamma = p(\bar{\Gamma})$$

is the shortest loop through v homotopic to γ

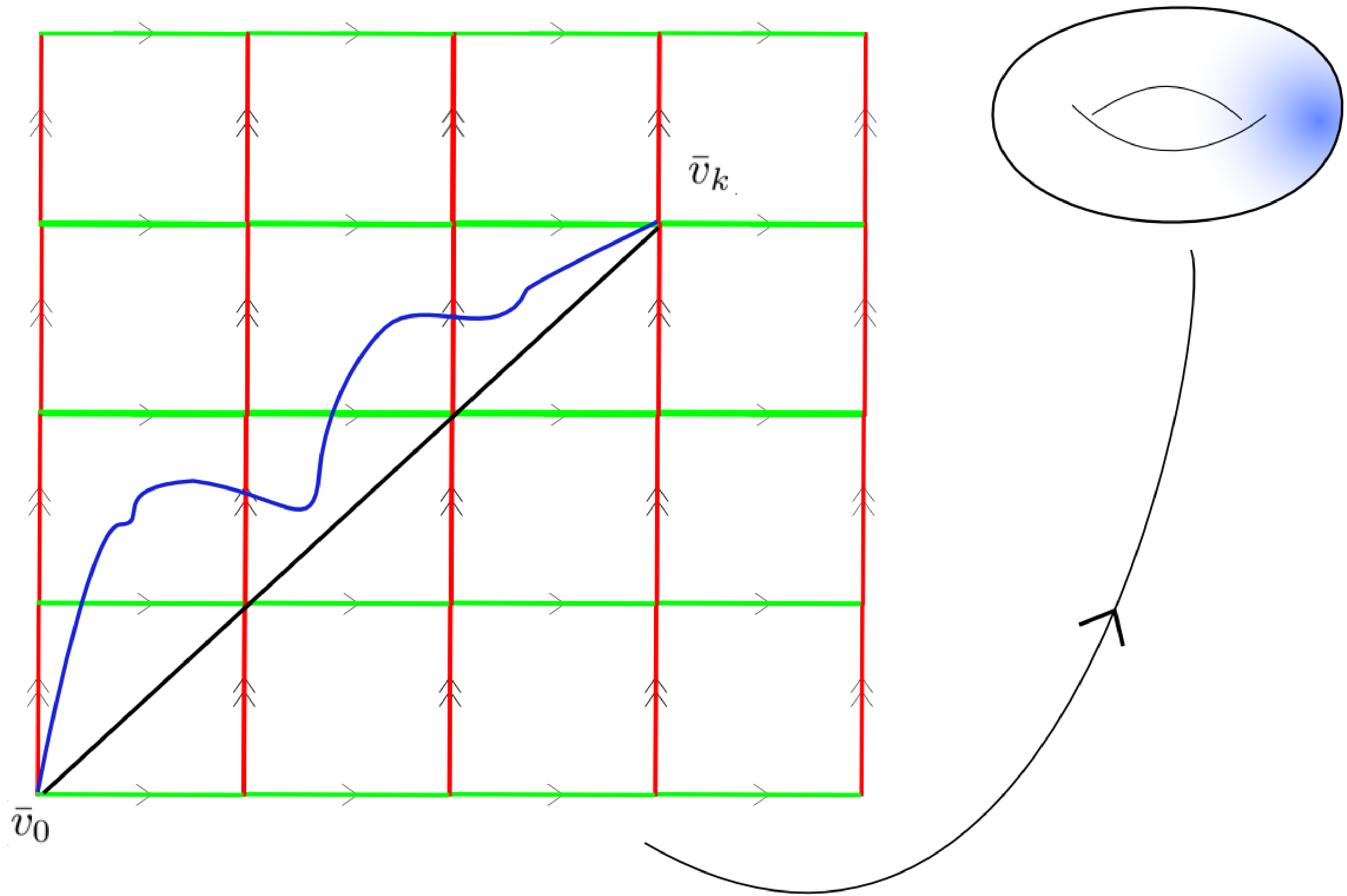
Example



Example



Example



The shortest in a given homotopy class

Input : A non-trivial loop γ on a mesh M
Output: the shortest loop on M homotopic to γ

The shortest in a given homotopy class

Input : A non-trivial loop γ on a mesh M
Output: the shortest loop on M homotopic to γ
For each vertex w on M do

The shortest in a given homotopy class

Input : A non-trivial loop γ on a mesh M
Output: the shortest loop on M homotopic to γ
For each vertex w on M do
    Find the nearest vertex v on γ;

The shortest in a given homotopy class

Input : A non-trivial loop γ on a mesh M
Output: the shortest loop on M homotopic to γ
For each vertex w on M do
    Find the nearest vertex v on γ;
    Find the shortest path connecting v and w, denoted as γ_0;

The shortest in a given homotopy class

Input : A non-trivial loop γ on a mesh M
Output: the shortest loop on M homotopic to γ
For each vertex w on M do
    Find the nearest vertex v on γ;
    Find the shortest path connecting v and w, denoted as γ_0;
    Construct the loop

$$\Gamma_w = \gamma_0 \gamma \gamma_0^{-1}$$

The shortest in a given homotopy class

Input : A non-trivial loop γ on a mesh M
Output: the shortest loop on M homotopic to γ
For each vertex w on M do
    Find the nearest vertex v on γ;
    Find the shortest path connecting v and w, denoted as γ_0;
    Construct the loop

$$\Gamma_w = \gamma_0 \gamma \gamma_0^{-1}$$

    Compute the shortest loop through w, homotopic to Γ_w

The shortest in a given homotopy class

Input : A non-trivial loop γ on a mesh M
Output: the shortest loop on M homotopic to γ
For each vertex w on M do
    Find the nearest vertex v on γ;
    Find the shortest path connecting v and w, denoted as γ_0;
    Construct the loop

$$\Gamma_w = \gamma_0 \gamma \gamma_0^{-1}$$

    Compute the shortest loop through w, homotopic to Γ_w
end

The shortest in a given homotopy class

Input : A non-trivial loop γ on a mesh M
Output: the shortest loop on M homotopic to γ
For each vertex w on M do
Find the nearest vertex v on γ;
Find the shortest path connecting v and w, denoted as γ_0;
Construct the loop

$$\Gamma_w = \gamma_0 \gamma \gamma_0^{-1}$$

Compute the shortest loop through w, homotopic to Γ_w
end
Select the loop with the minimal length

$$\Gamma = \min_{w \in M} \Gamma_w$$

References

Algorithms presented here can be found in :

D. Gu and S. Yau, Computational conformal geometry. Somerville, Mass, USA: International Press, 2008.