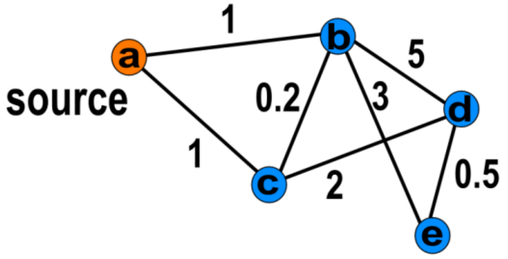
Graphs Algorithms

Mustafa Hajij



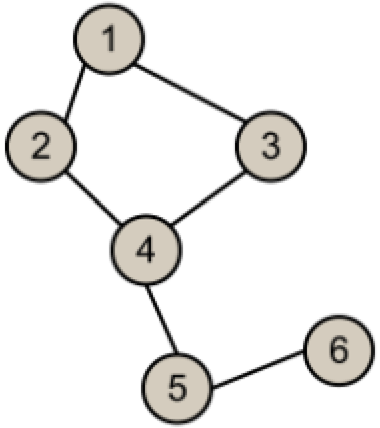
Graphs

A **graph** is an ordered pair (V,E) where,

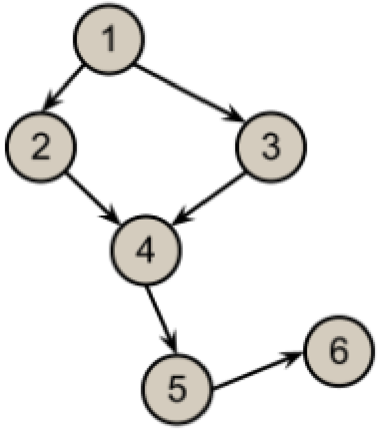
• V is the *vertex set (also node set)* whose elements are the vertices, or *nodes* of the graph.

• E is the *edge set* whose elements are the edges, or connections between vertices, of the graph. If the graph is undirected, individual edges are unordered pairs.

• If the graph is directed, edges are ordered pairs

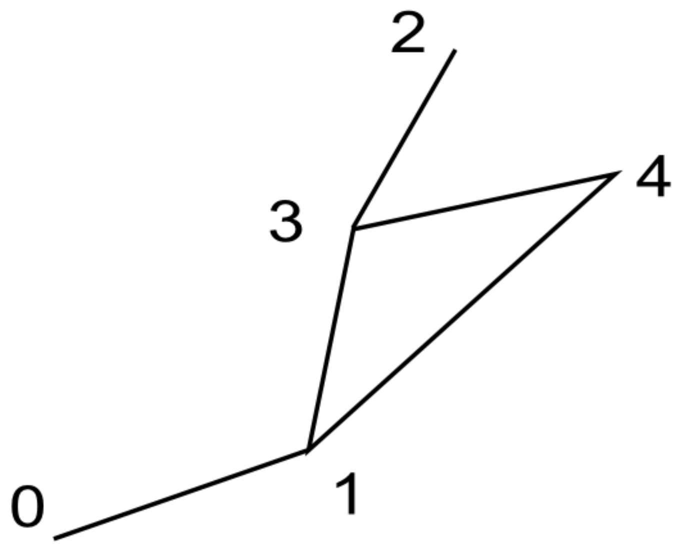


undirected



directed

Graphs representation: list of nodes and edges

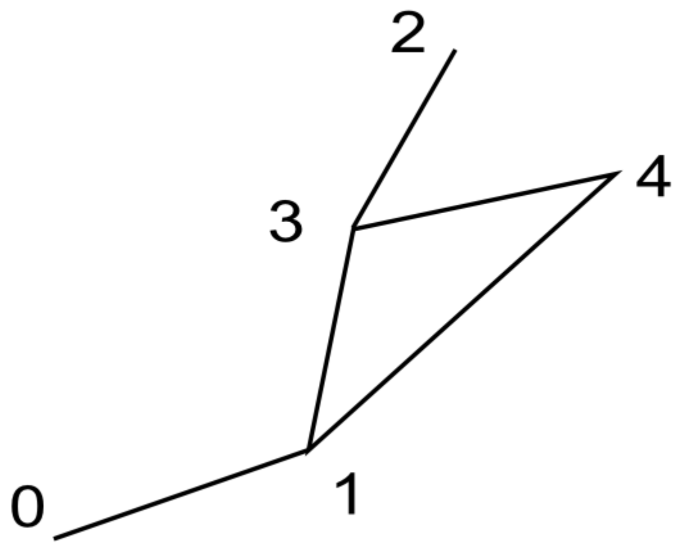


Nodes: [0,1,2,3,4]

Edges: [[0,1], [1,3],[1,4] [3,4], [3,2]]

Note that if the graph is connected, then the list of edges is enough to determine the graph completely.

Graphs representation: list of nodes and edges

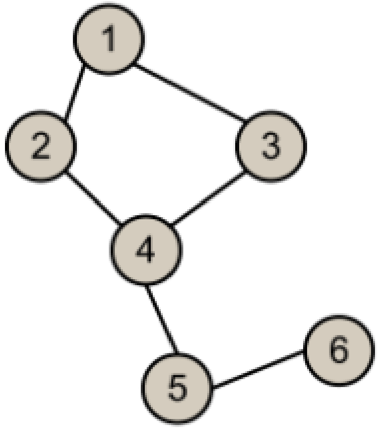


Nodes: [0,1,2,3,4]

Edges: [[0,1], [1,3],[1,4] [3,4], [3,2]]

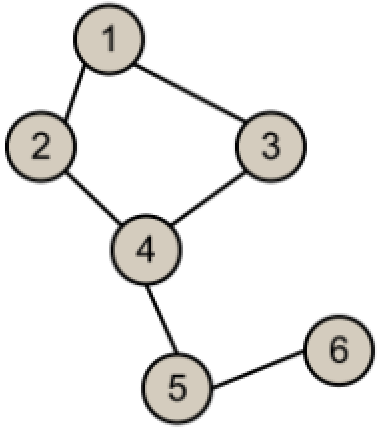
The order of the vertices is important only if the graph is directed.

Graphs representation: adjacency matrix

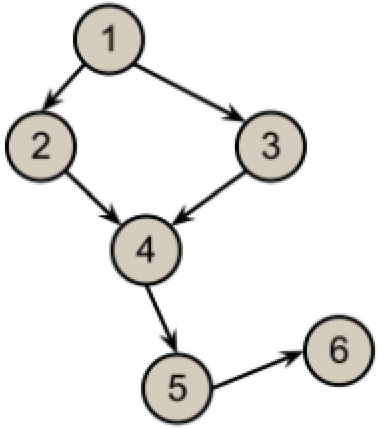


	1	2	3	4	5	6
1	0	1	1	0	0	0
2	1	0	0	1	0	0
3	1	0	0	1	0	0
4	0	1	1	0	1	0
5	0	0	0	1	0	1
6	0	0	0	0	1	0

Graphs representation: adjacency matrix



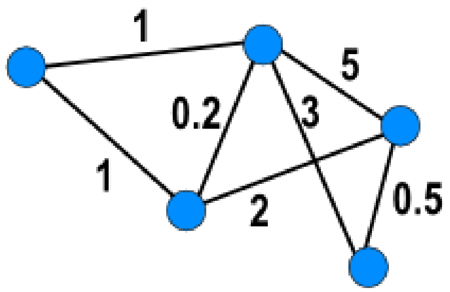
	1	2	3	4	5	6
1	0	1	1	0	0	0
2	1	0	0	1	0	0
3	1	0	0	1	0	0
4	0	1	1	0	1	0
5	0	0	0	1	0	1
6	0	0	0	0	1	0



	1	2	3	4	5	6
1	0	1	1	0	0	0
2	-1	0	0	1	0	0
3	-1	0	0	1	0	0
4	0	-1	-1	0	1	0
5	0	0	0	-1	0	1
6	0	0	0	0	-1	0

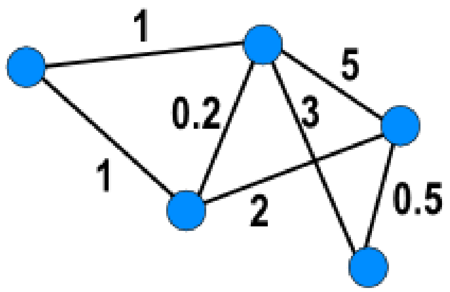
Weighted Graphs

A weighted graph is a graph in which every edge has a weight (non-negative real number)



Weighted Graphs

A weighted graph is a graph in which every edge has a weight (non-negative real number)



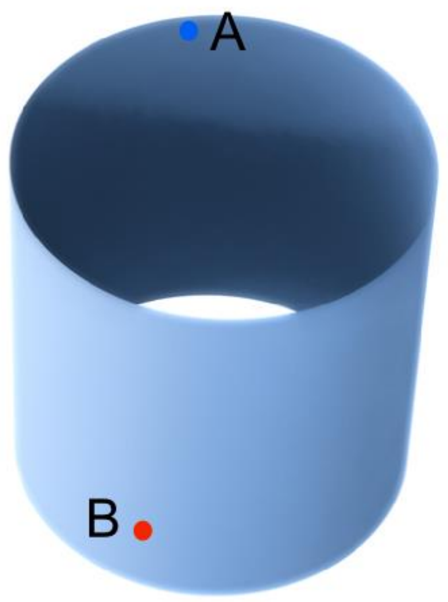
Formally speaking :

A weight function $w: E \rightarrow R^{+}$. In other words, the function w associates to every edge e a positive number (weight) $w(e)$

A weighted graph is a graph G=(V,E) with a weight function $w: E \rightarrow R^{+}$.

Shortest distance

What is the shortest distance between A and B?



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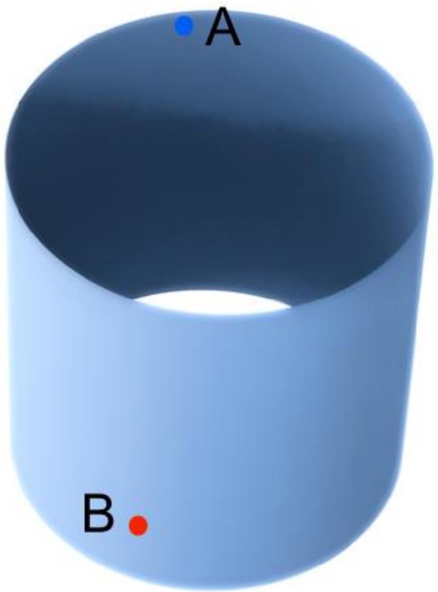
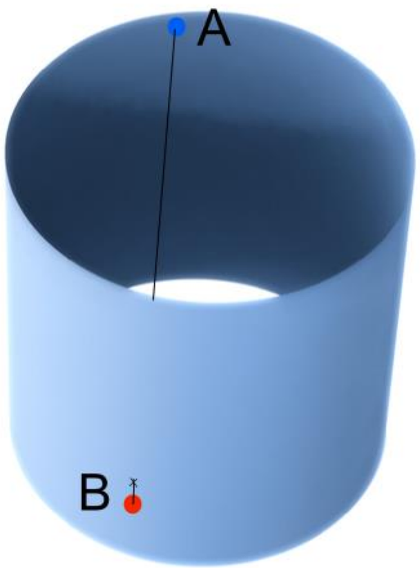
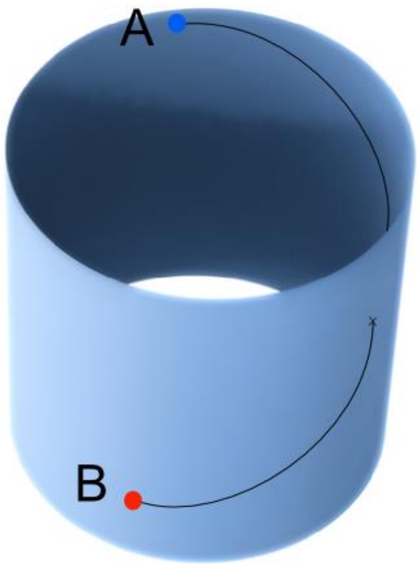




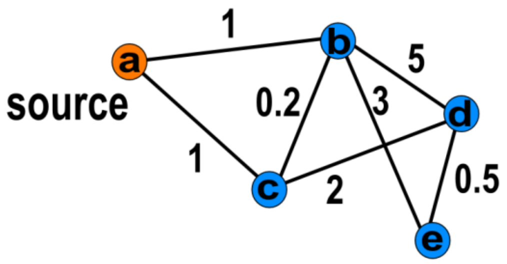


Dijkstra algorithm

The basic Dijkstra algorithm operates on connected, undirected, weighted graph.

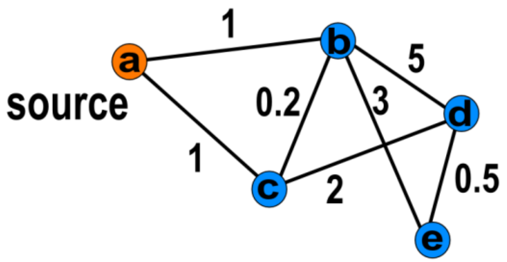
```
Dijkstra(Graph, source):
    for each vertex v in Graph:
        distance[v] := infinity            // the initial distance from source to any other vertex v is infinity
        previous[v] := undefined
    distance[source] := 0                  // Distance from the source to itself is zero
    Q := the set of all nodes in the Graph // all nodes are going in this container
    while Q is not empty:                  // main loop
        u := the node in Q with smallest distance from the source   (what kind of queue you use here?)
        remove u from Q                    //the source will be removed first
        for each neighbor v of u:          // v is still in the container Q
            alt := distance[u] + length(u, v)
            if alt < distance[v]           //A shorter path from v to the source has been found
                distance[v] := alt
                previous[v] := u
    return distance[], previous[ ]
```

Dijkstra algorithm : Example

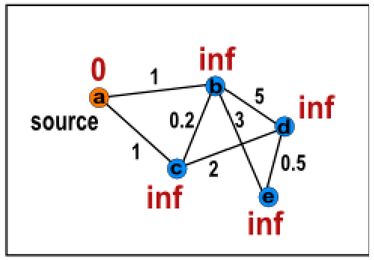


Input : weighted graph
with a source vertex

Dijkstra algorithm : Example

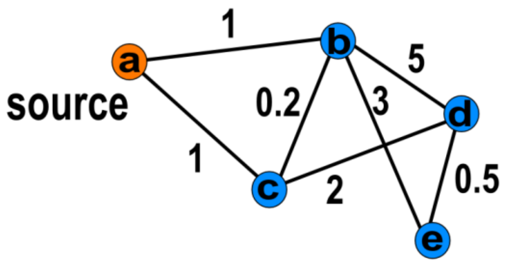


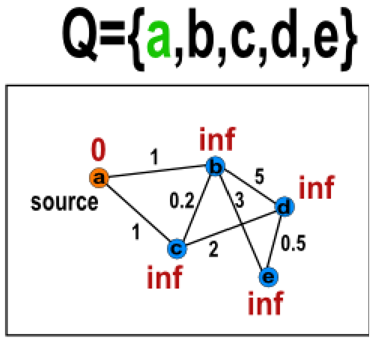
Q={a,b,c,d,e}



Algorithm starts by initializing the distance to every vertex other than the source to infinity. We also create a queue Q and put in it all vertices of G.

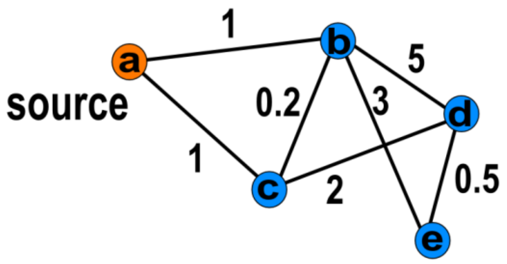
Dijkstra algorithm : Example

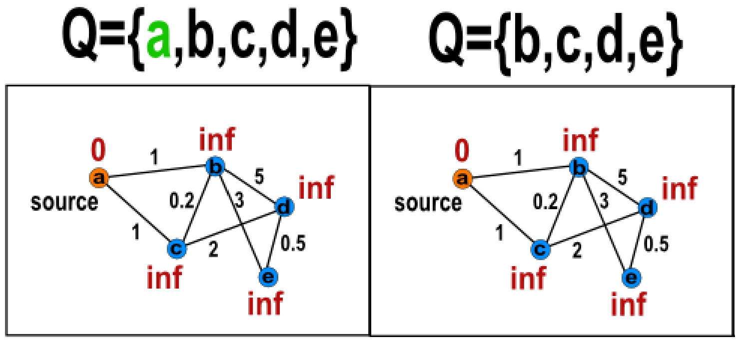




When we enter the while loop we dequeue the element in Q with shortest distance to source. In this case it is a.

Dijkstra algorithm : Example





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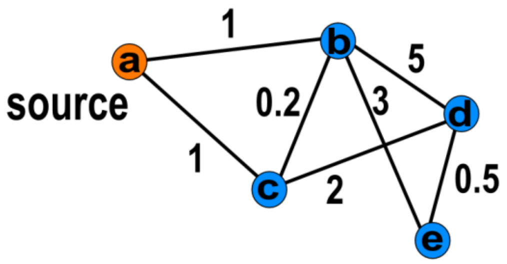
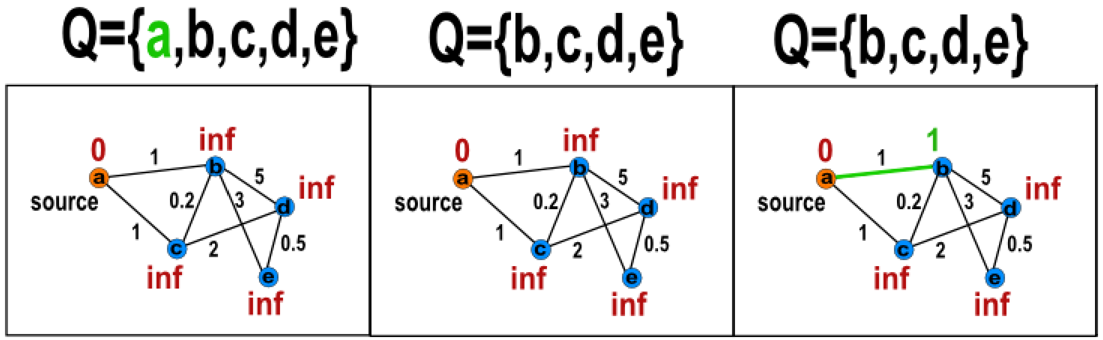
Dijkstra algorithm : Example





Now we visit all neighbors of a and update the distance to them:

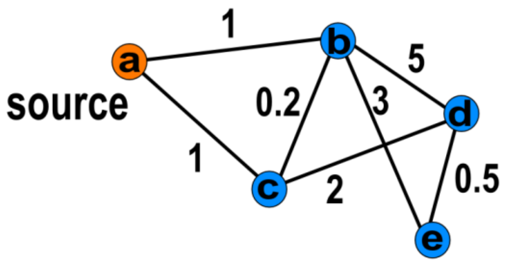
for each neighbor v of u:

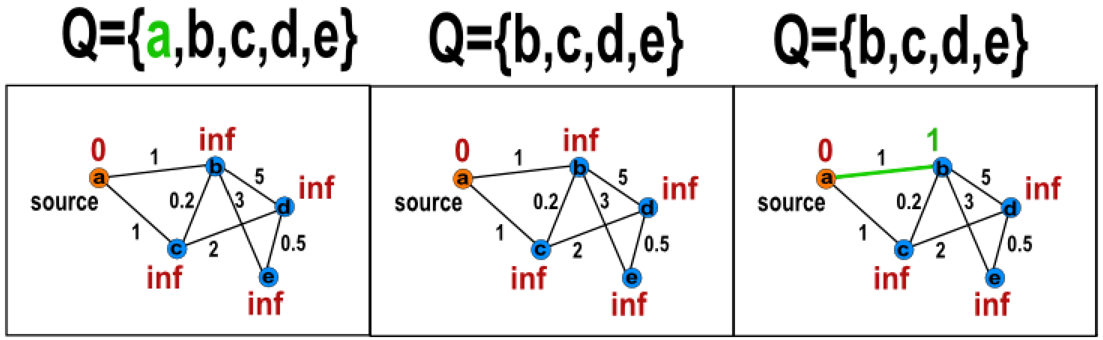
alt := distance[u] + length(u, v)

if alt < distance[v]

distance[v] := alt

Dijkstra algorithm : Example



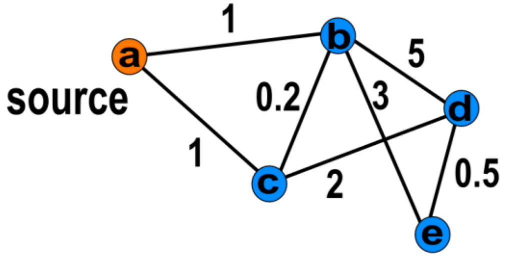


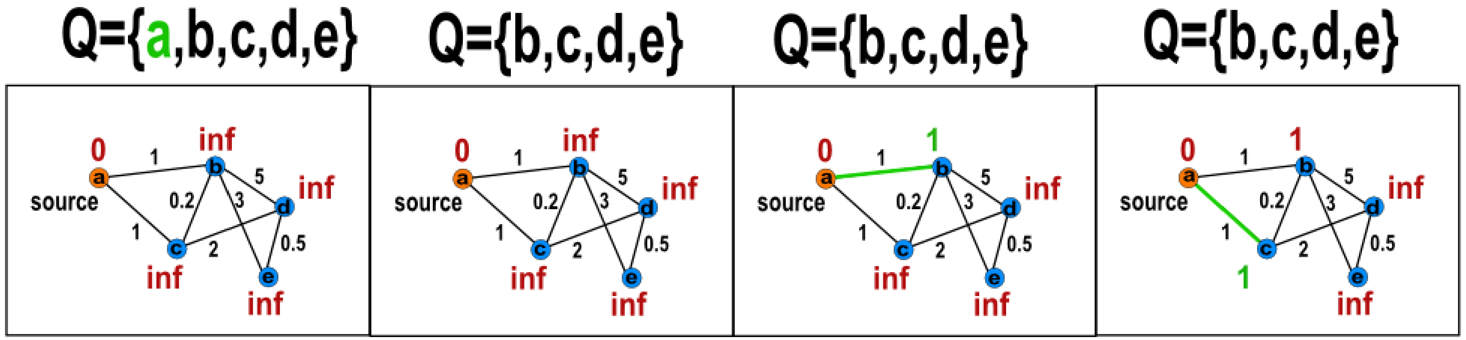
In this case we update the distance to b to 1.

Now we visit all neighbors of a and update the distance to them:

for each neighbor v of u:
alt := distance[u] + length(u, v)
if alt < distance[v]
distance[v] := alt

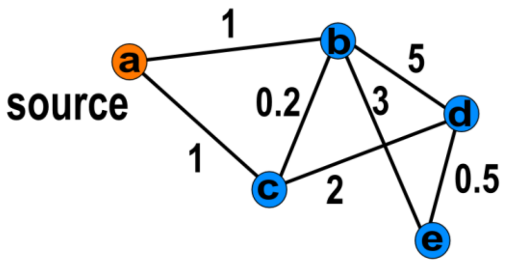
Dijkstra algorithm : Example

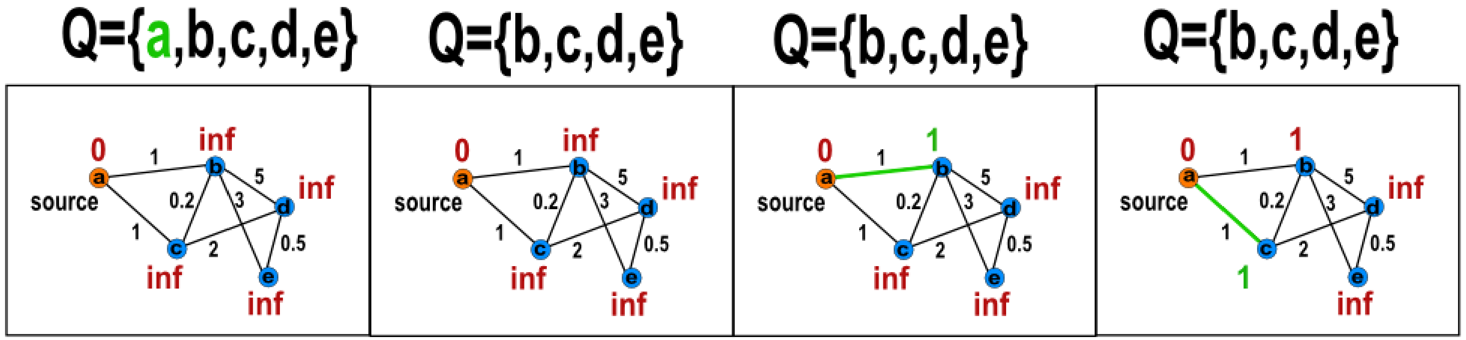




Here we update the distance to c to be 1 as well

Dijkstra algorithm : Example

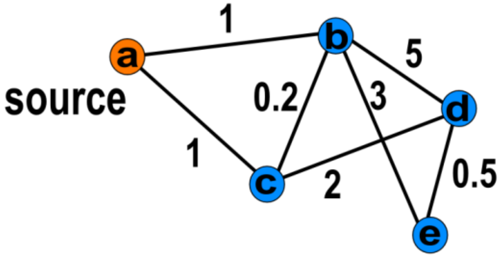




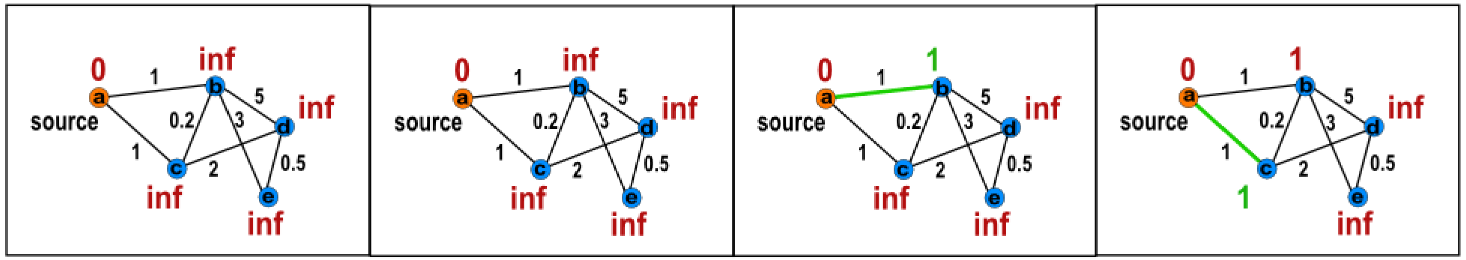
Here we update the distance to c to be 1 as well

At this stage all neighbors of a have been visited so we check the queue again: if is not empty we start the process again.

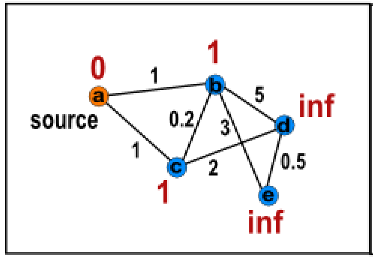
Dijkstra algorithm : Example



Q={a,b,c,d,e} Q={b,c,d,e} Q={b,c,d,e} Q={b,c,d,e}

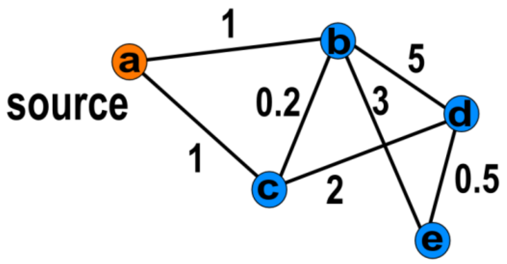


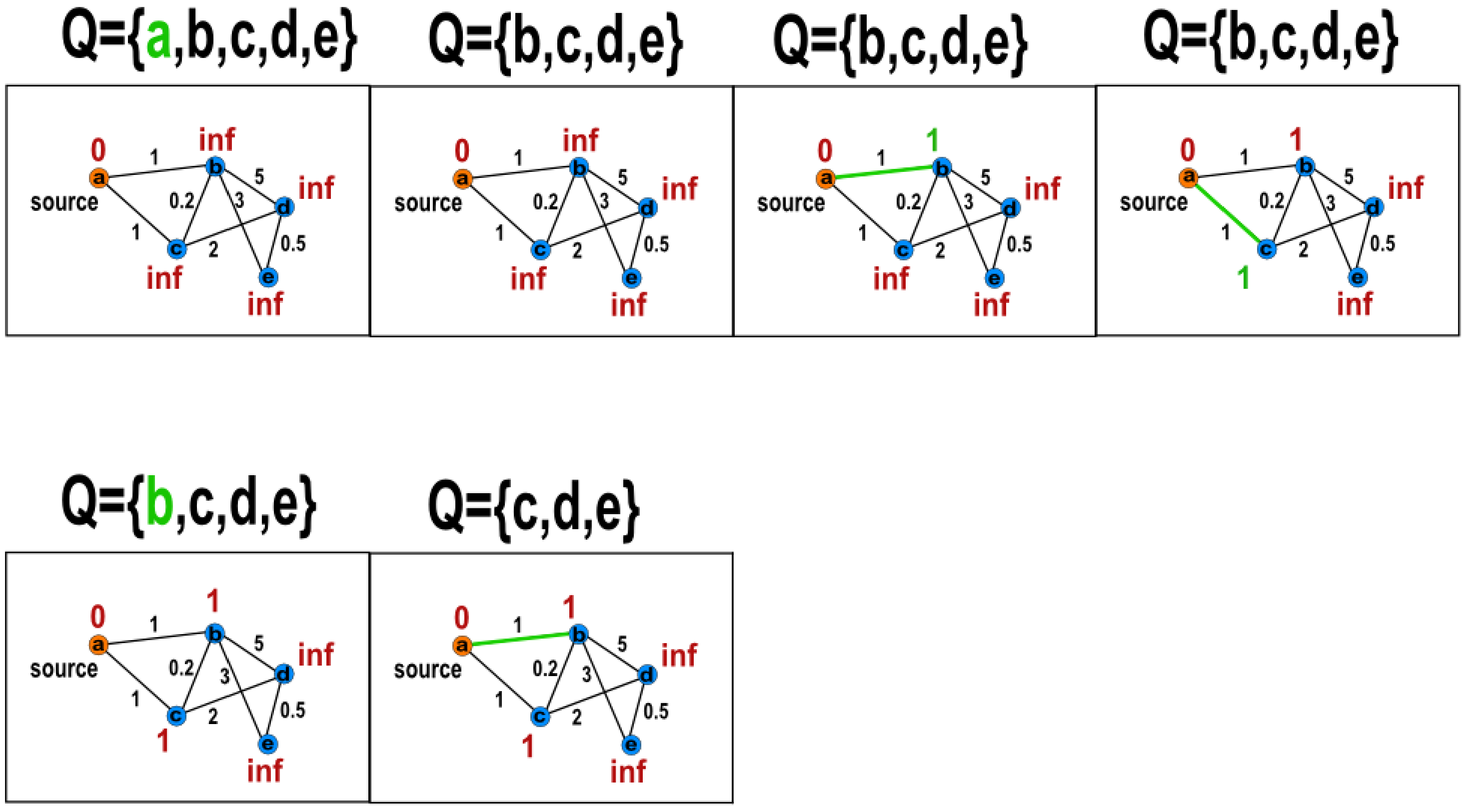
Q={b,c,d,e}



We select b (the closest element to a so far)

Dijkstra algorithm : Example

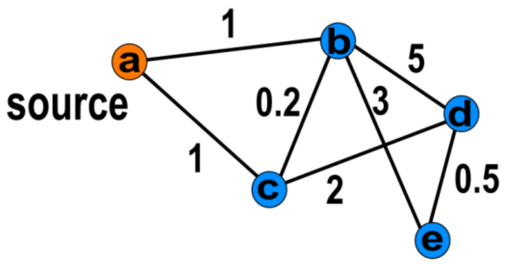


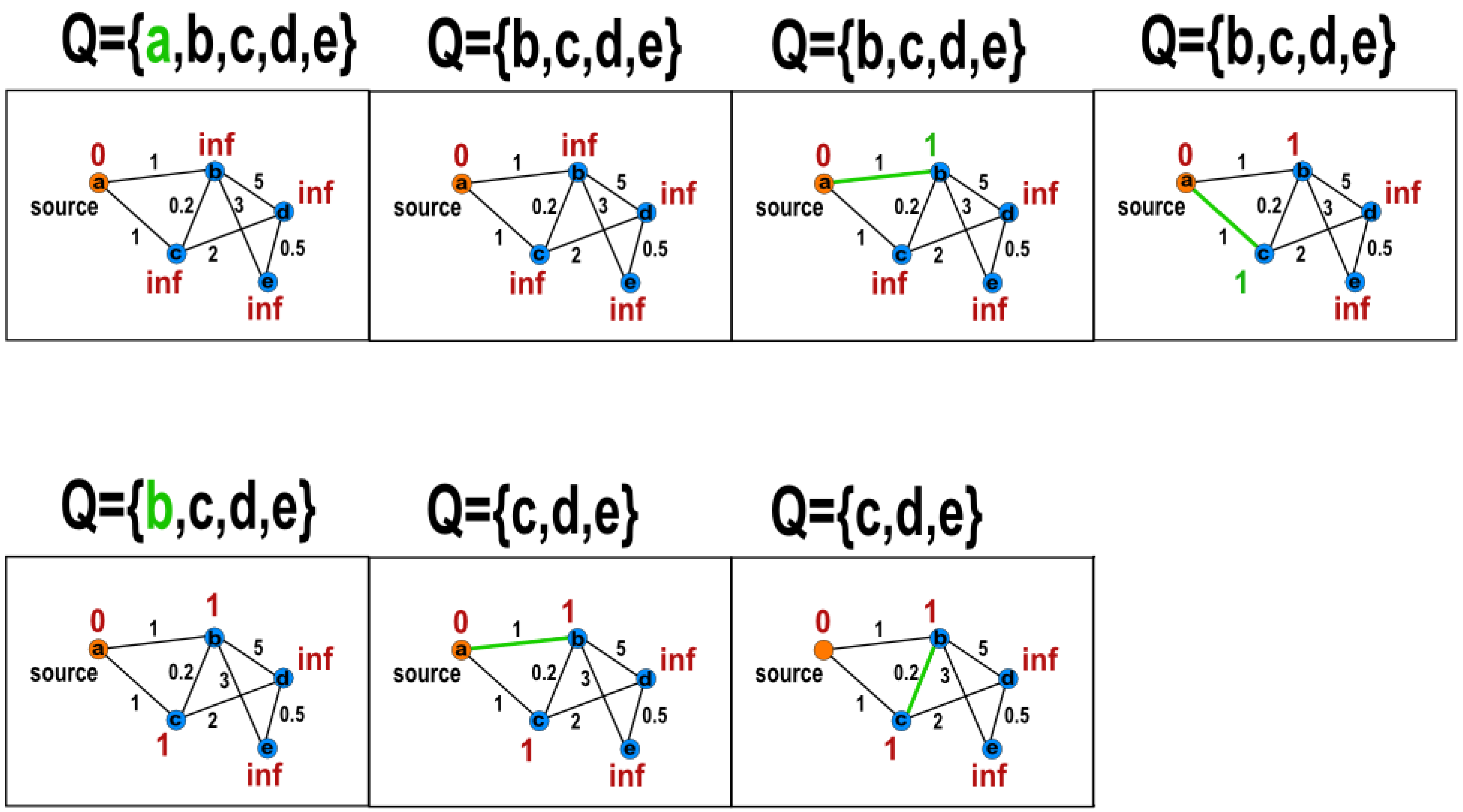


Remove b from the queue and start visiting all its neighbors and update the distance.

In this case the distance does not update—why ?

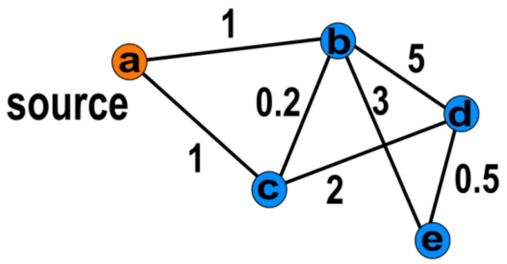
Dijkstra algorithm : Example

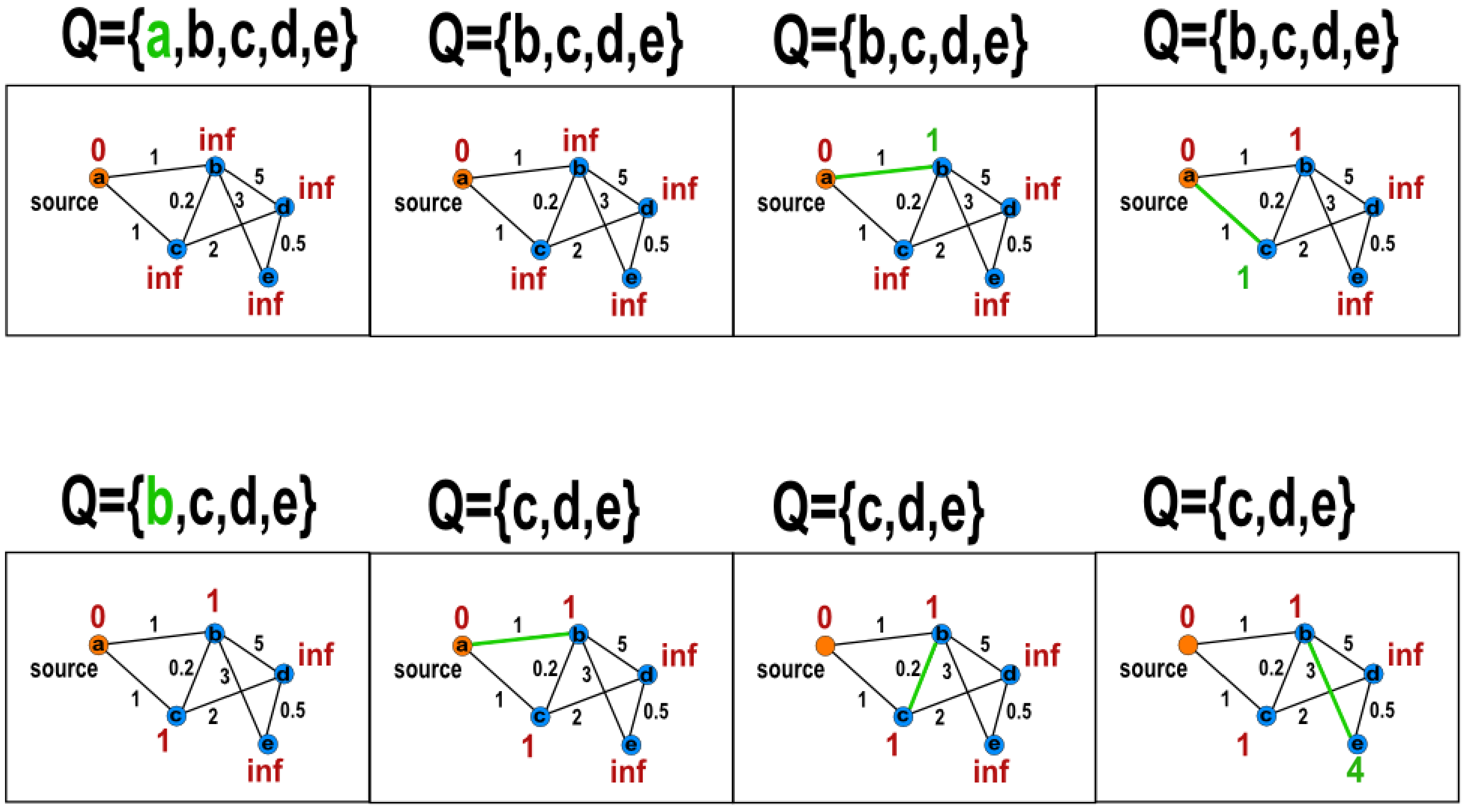




The distance here also does not update

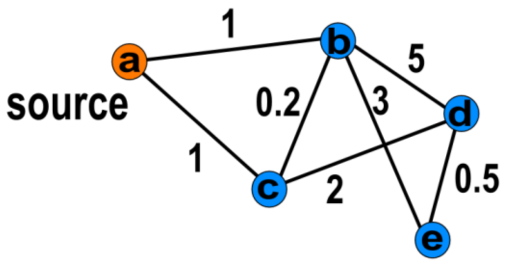
Dijkstra algorithm : Example



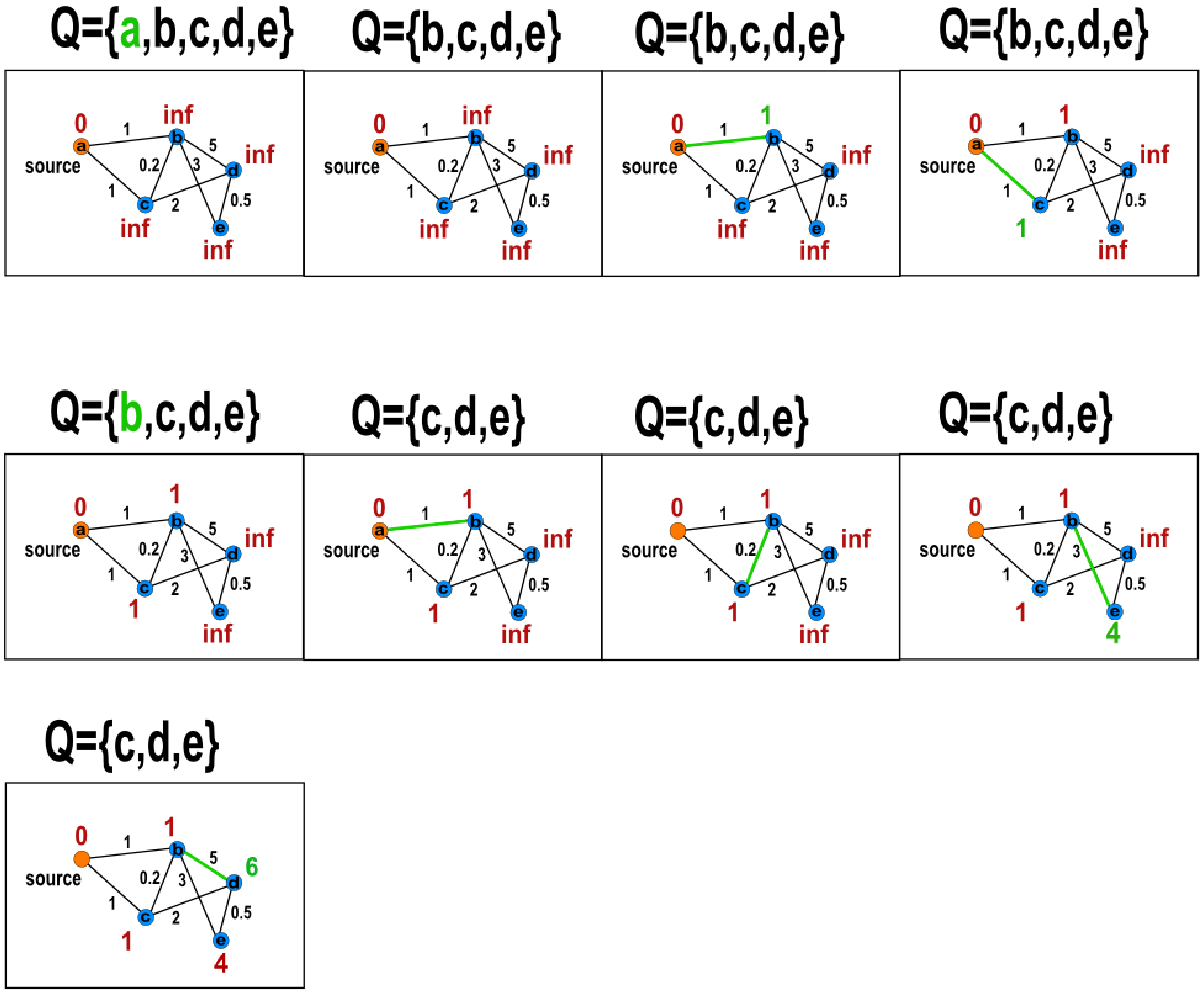


Here we update the distance from a to e to be 4

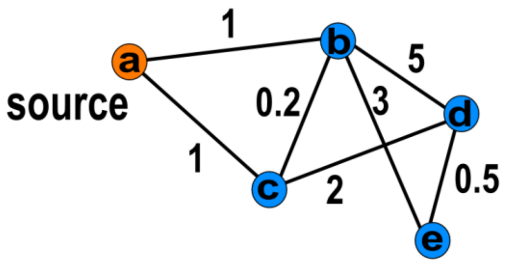
Dijkstra algorithm : Example



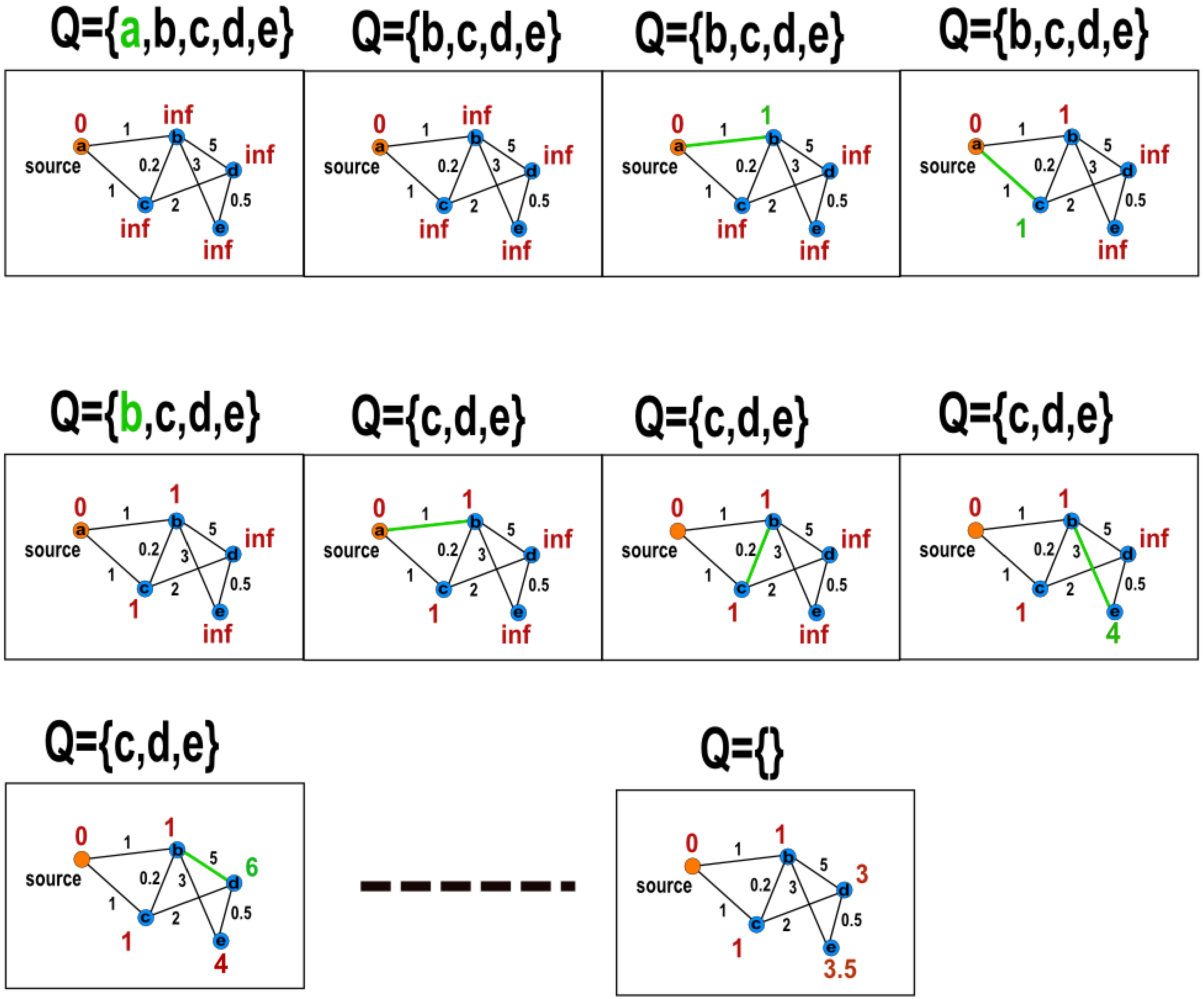
Update the distance from a to d



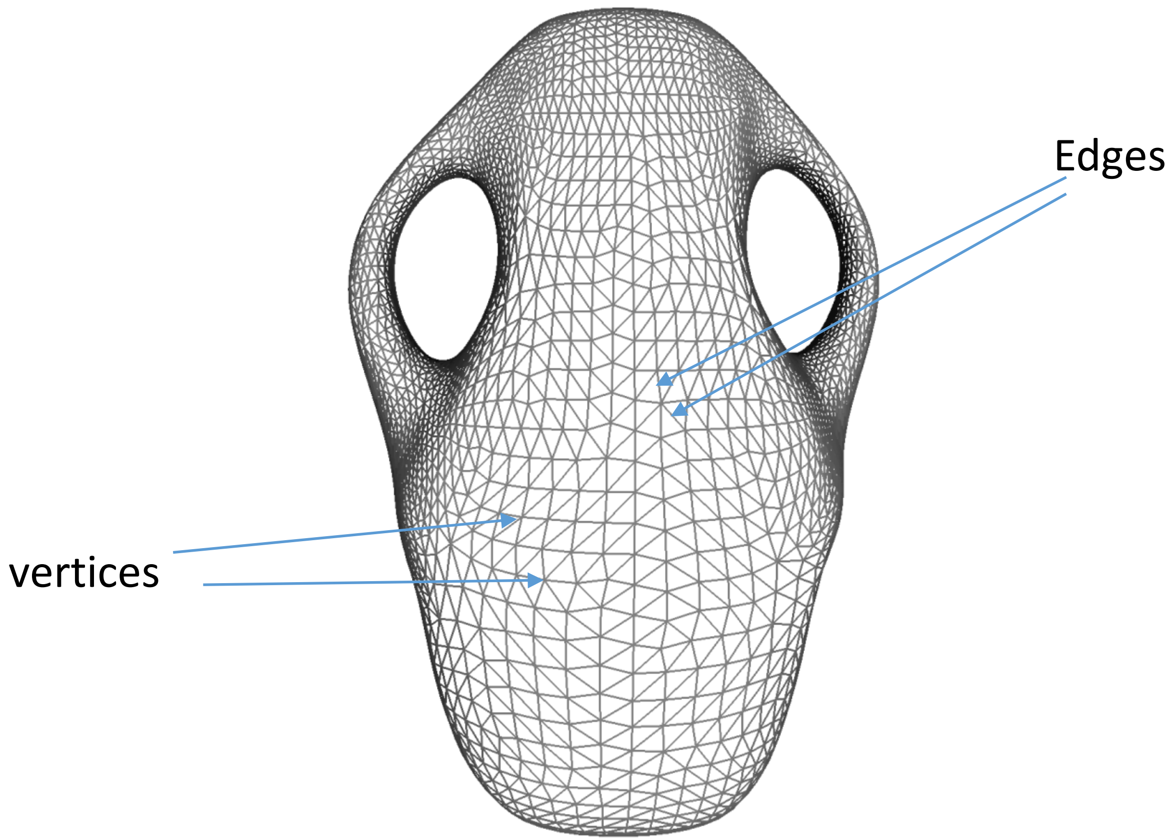
Dijkstra algorithm : Example



And so on

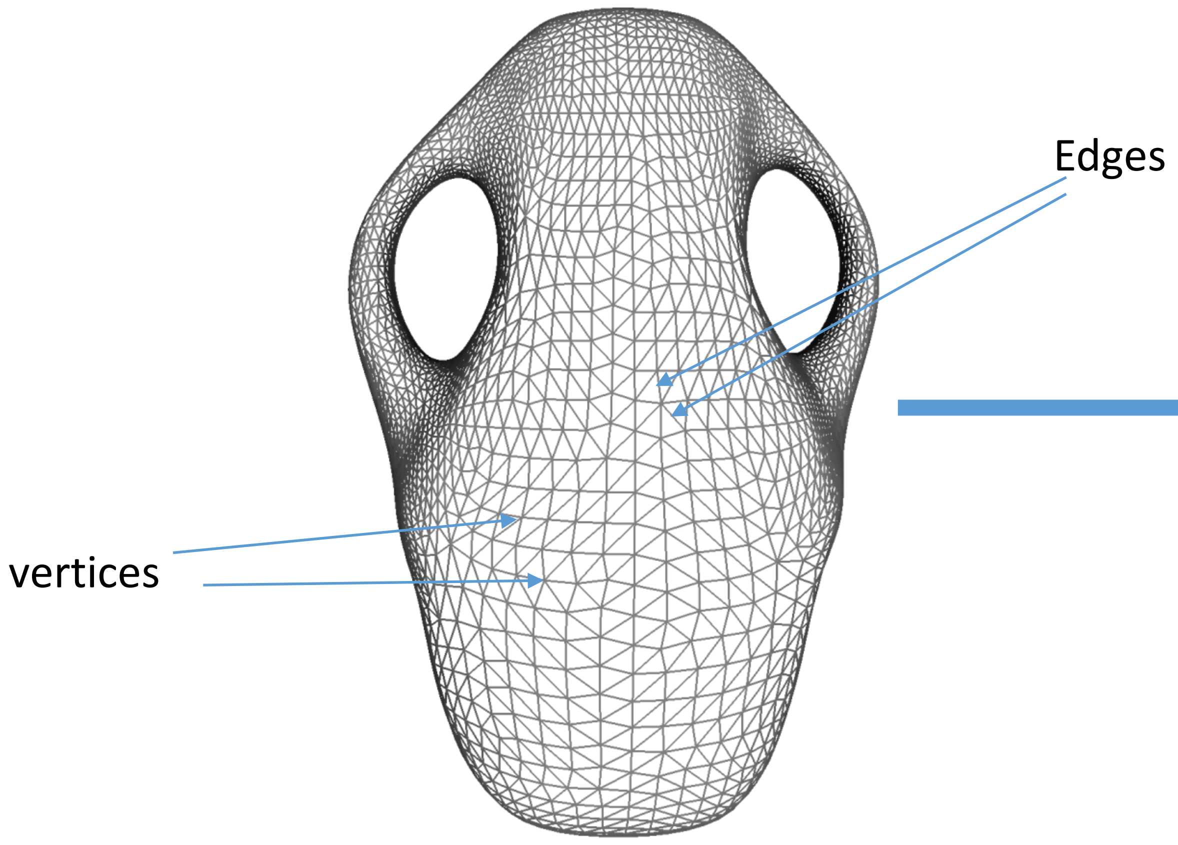


Example



We can view a mesh as a graph and apply Dijkstra algorithm on it.

Example



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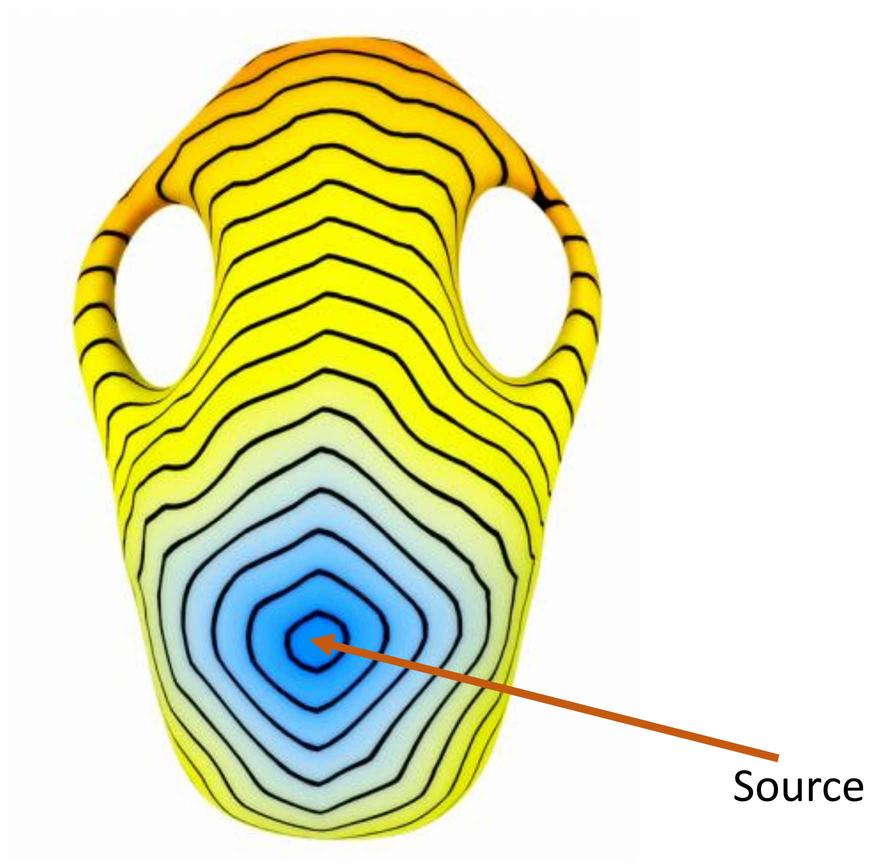
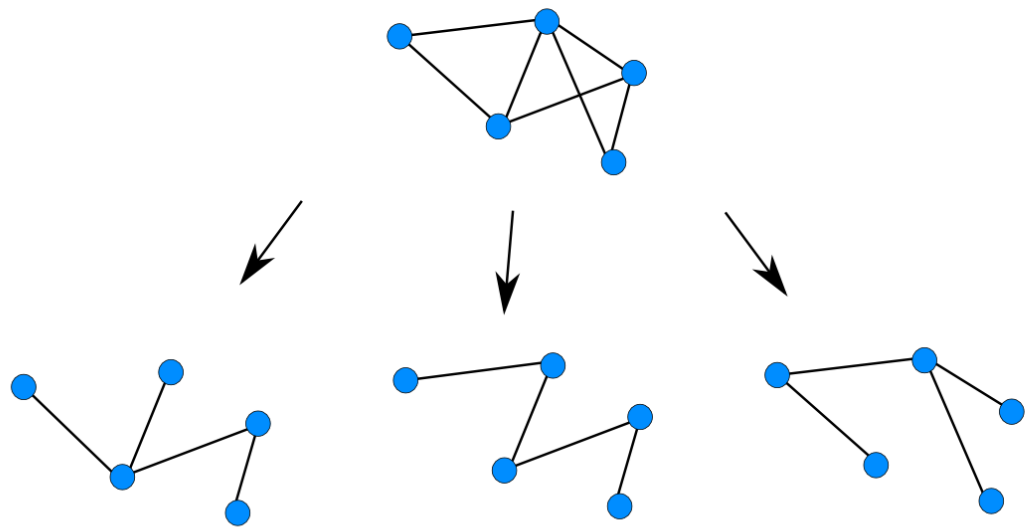


Blue indicates the regions closest to the source

Spanning Tree

Let $G = (V, E)$ be a connected weighted graph. A spanning tree for G is a subgraph of G which includes all of the vertices of G and is a tree.

A graph might have more than one spanning tree

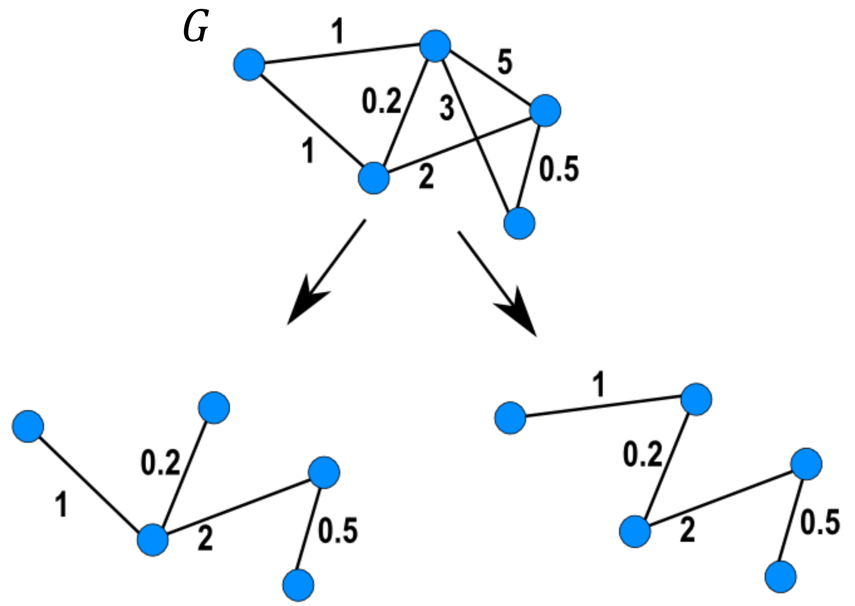


Spanning trees for G

Minimal Spanning Tree

Let $G = (V, E, w)$ be a connected weighted graph. A minimal spanning tree for G is a spanning tree whose sum of edge weights is as small as possible.

A graph might have more than one minimal spanning tree. However, if all edges in the graph have unique weights then the minimal spanning tree is unique.



Minimal spanning trees for G

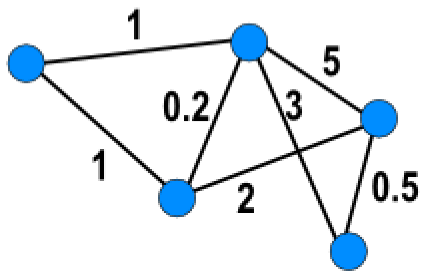
Kruskal's Algorithm

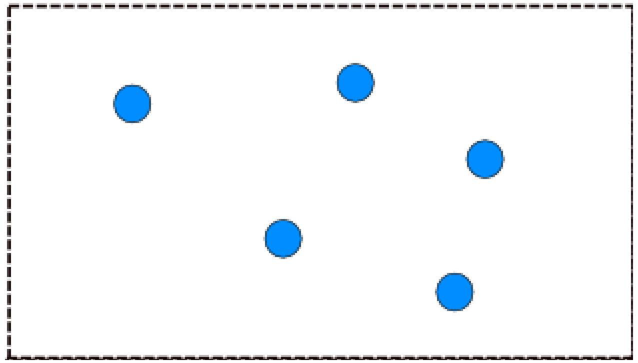
Let $G = (V, E, w)$ be a connected weighted graph. The Kruskal's algorithm is a greedy algorithm.

Informally, the algorithm can be given by the following three steps :

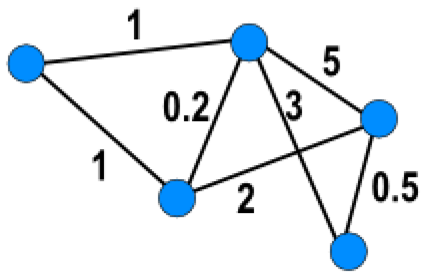
1. Set V_T $to\ be\ V, Set\ E_T = \{\}$. Let $S = E$
2. While S is not empty and T is not a spanning tree
    1. Select an edge e from S with the minimum weight and delete e from S.
    2. If e connects two separate trees of T then add e to E_T

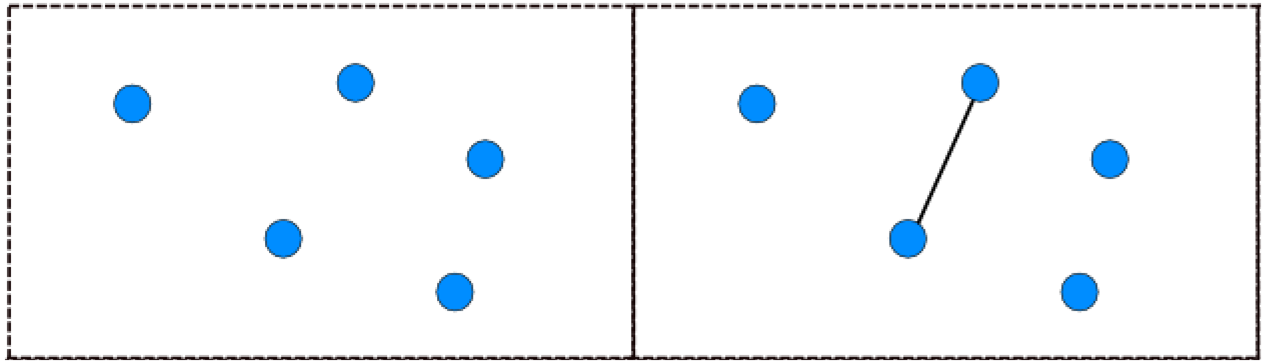
Kruskal’s Algorithm Example



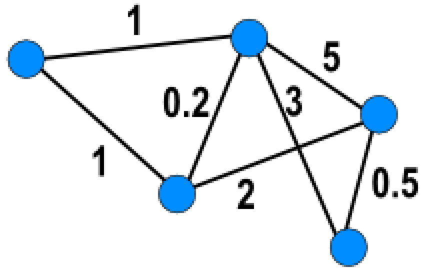


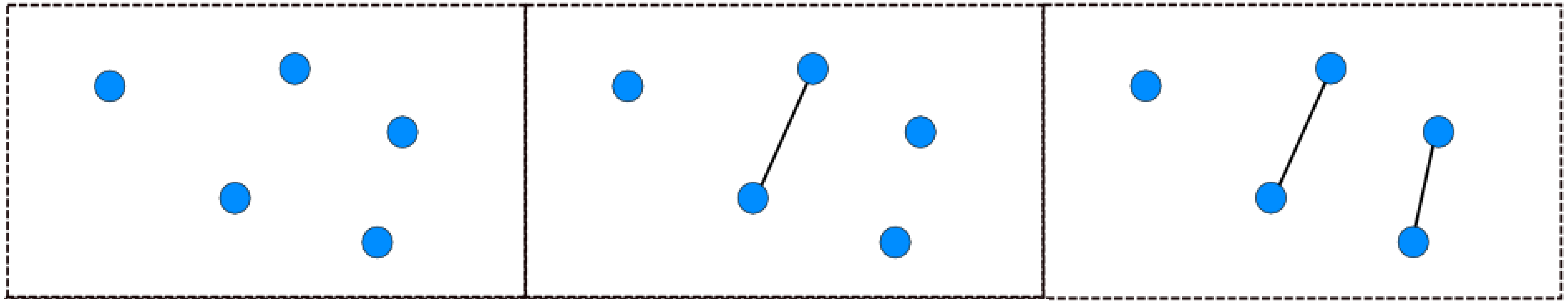
Kruskal’s Algorithm Example



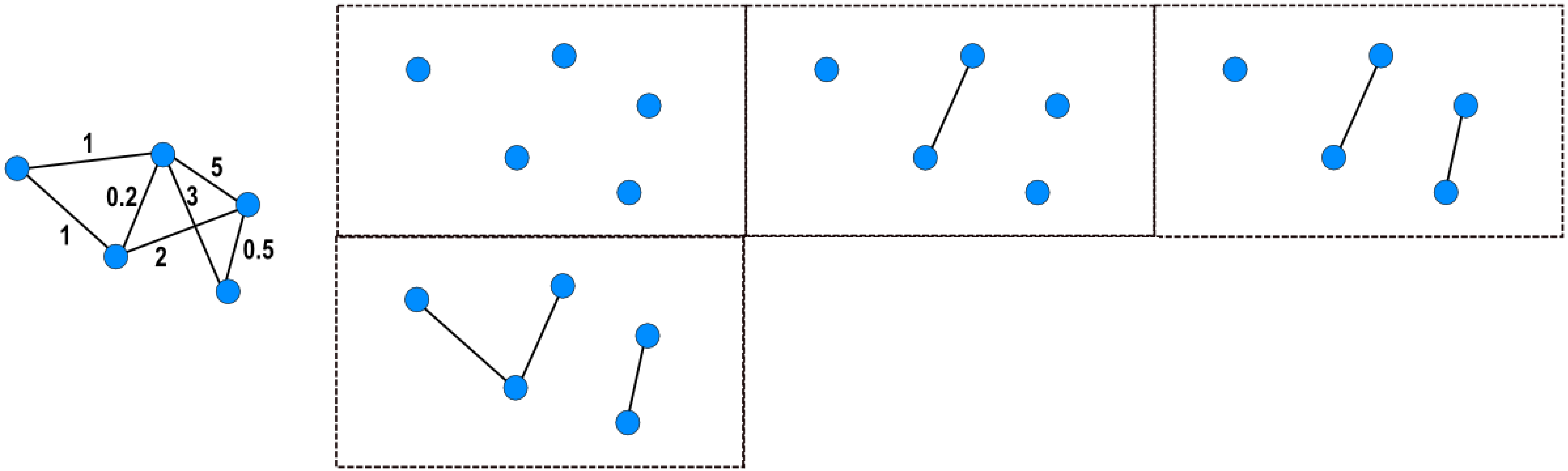


Kruskal’s Algorithm Example

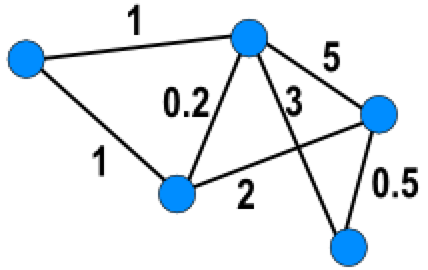


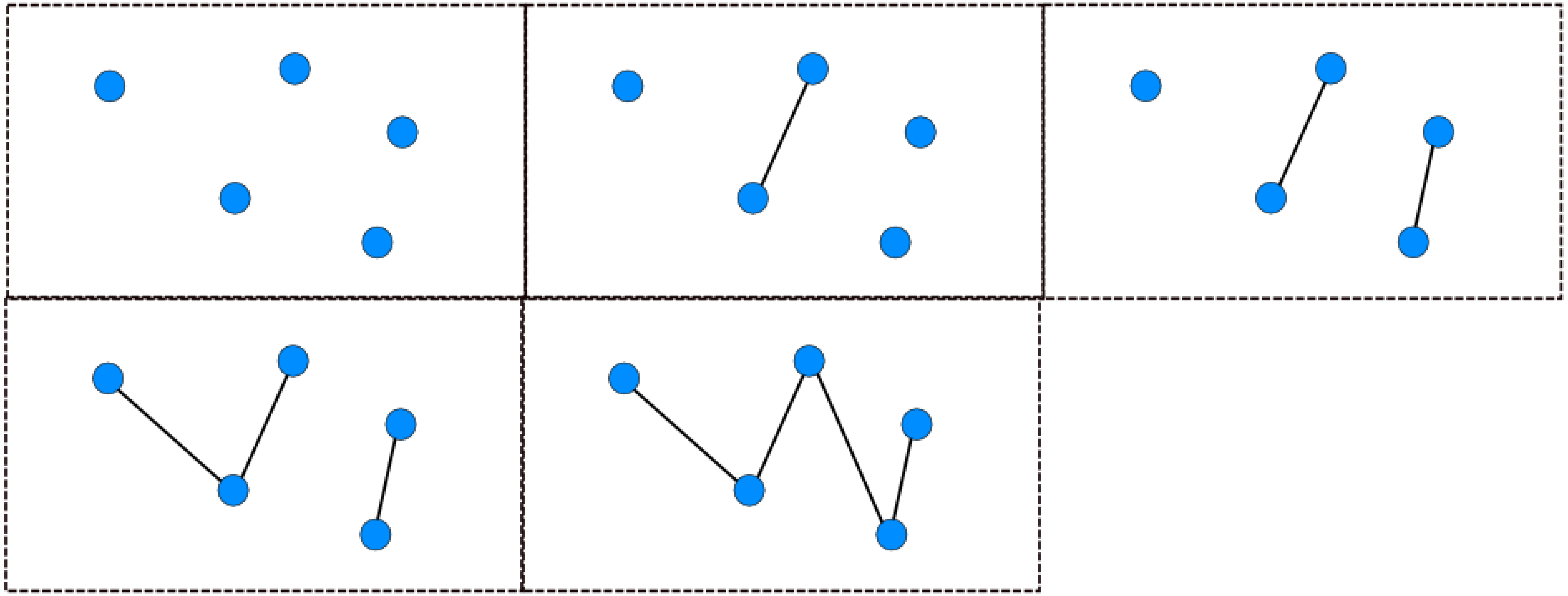


Kruskal's Algorithm Example



Kruskal’s Algorithm Example





Kruskal's Algorithm

Let $G = (V, E, w)$ be a connected weighted graph. The Kruskal's algorithm is a greedy algorithm

This can be implemented using union-find data-structure

1- A= {}
2-foreach $v \in V$:
3- MAKE-SET(v)
4-foreach (u, v) in E ordered by weight(u, v), increasing:
5- if FIND-SET(u) ≠ FIND-SET(v):
6- $A = A \cup \{(u, v)\}$
7- UNION(u, v)
8-return A

Prim's Algorithm

Let $G = (V, E, w)$ be a connected weighted graph. The Prim's algorithm is a greedy algorithm

Informally, the algorithm can be given by the following three steps :

1. Select an arbitrary vertex v from V. Set $V_T = \{v\}$ and $E_T = \{\quad\}$
2. Grow the tree by one edge : choose an edge e(u,v) from the set E with the lowest cost such that u in V_T and v is in $V \backslash V_T$ then add v to V_T and add e to E_T
3. If $V_T = V$ break, otherwise go to step 2.