# Graph Laplacian

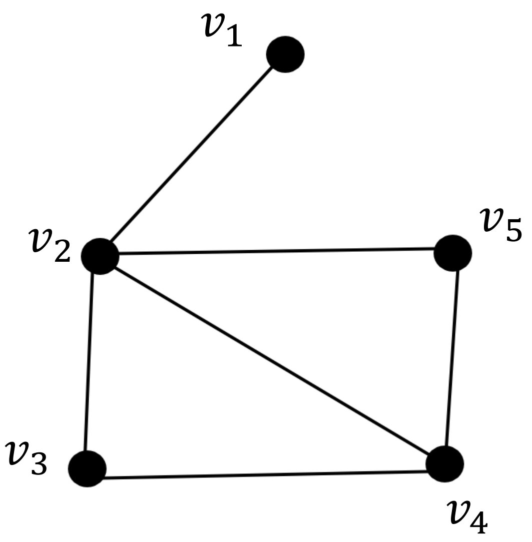
Mustafa Hajij

# Graph Laplacian

Let G be a graph on n nodes. The Graph Laplacian is an n by n matrix given by :

$$L = D - A$$

Where D is the degree matrix and A is the adjacency matrix




$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

# Symmetric Graph Laplacian

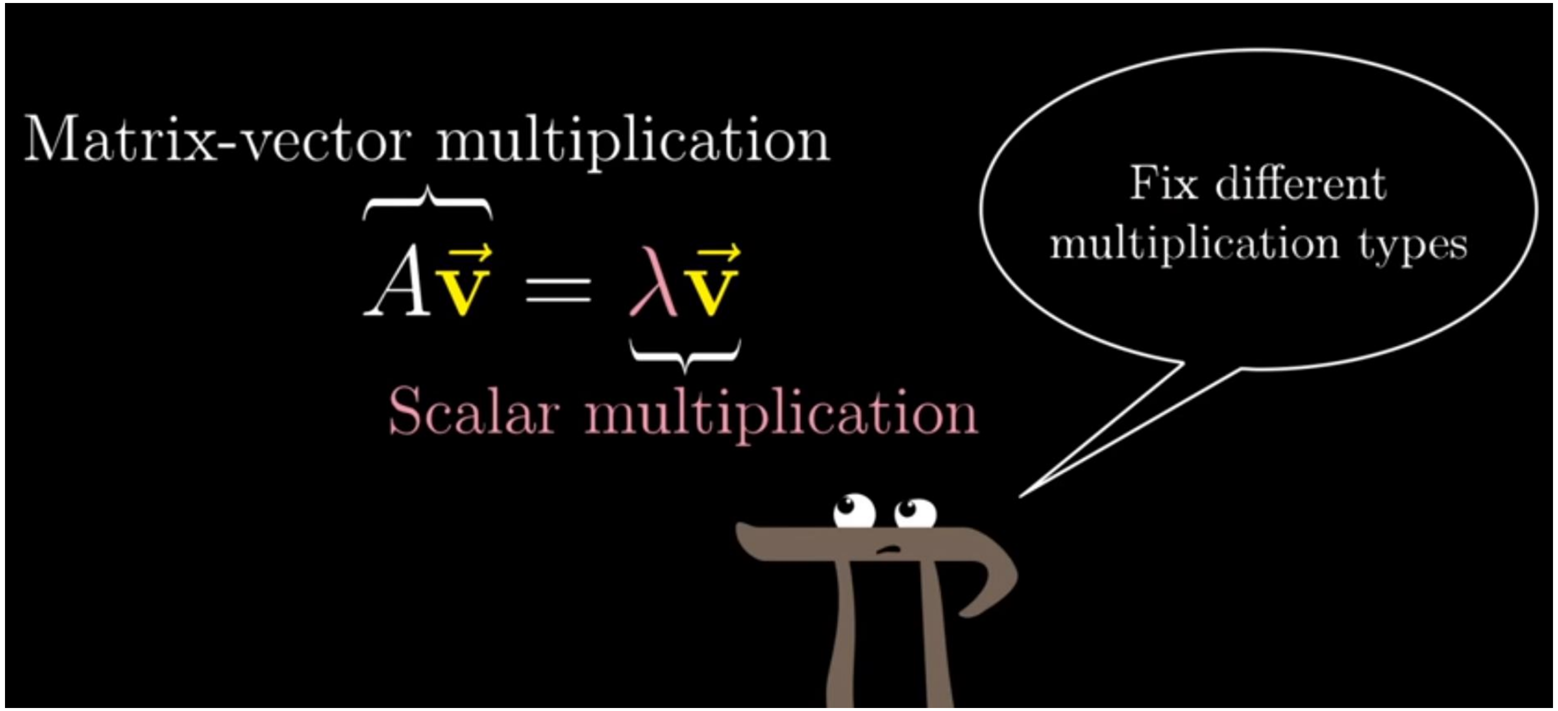
$$L^{\text{sym}} := D^{-1/2} L D^{-1/2} = I - D^{-1/2} A D^{-1/2},$$

Explicitly this is given by:

$$L_{i,j}^{\text{sym}} := \begin{cases} 1 & \text{if } i = j \text{ and } \deg(v_i) \neq 0 \\ -\frac{1}{\sqrt{\deg(v_i)\deg(v_j)}} & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

# Eigenvalues and Eignenvector of a matrix

Watch this lecture for [review](review)



# Eigenvalues and Eignenvector of a matrix

Symmetric Matrix → 

Eigenvalues are real

Eigenvectors are orthogonal

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If $A \in R^{n \times n}$ is a symmetric matrix then it has an orthonormal set of eigenvectors $u_1, u_2, \ldots, u_n$ corresponding to (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

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The graph Laplacian is a symmetric matrix

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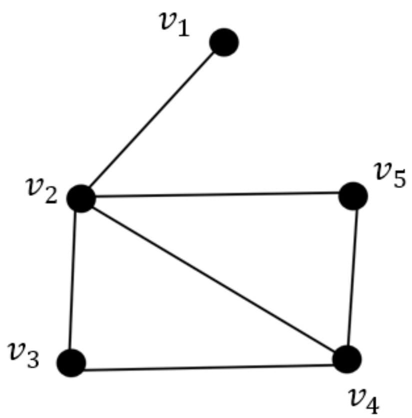

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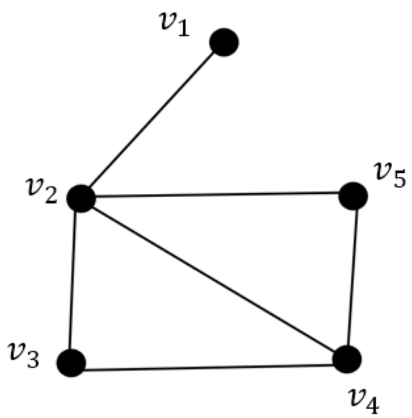


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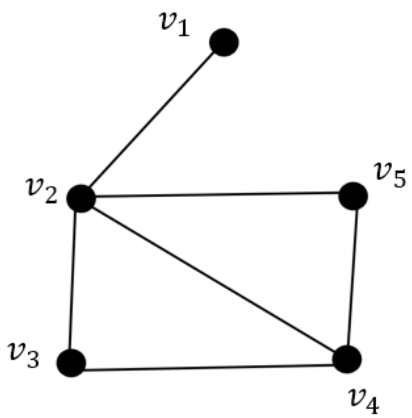


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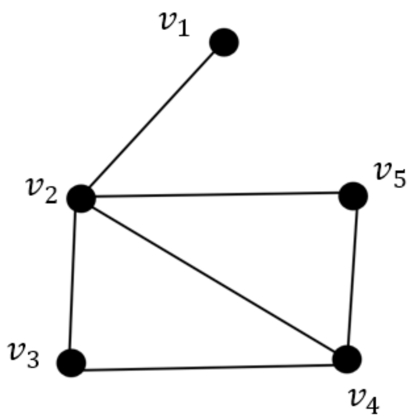


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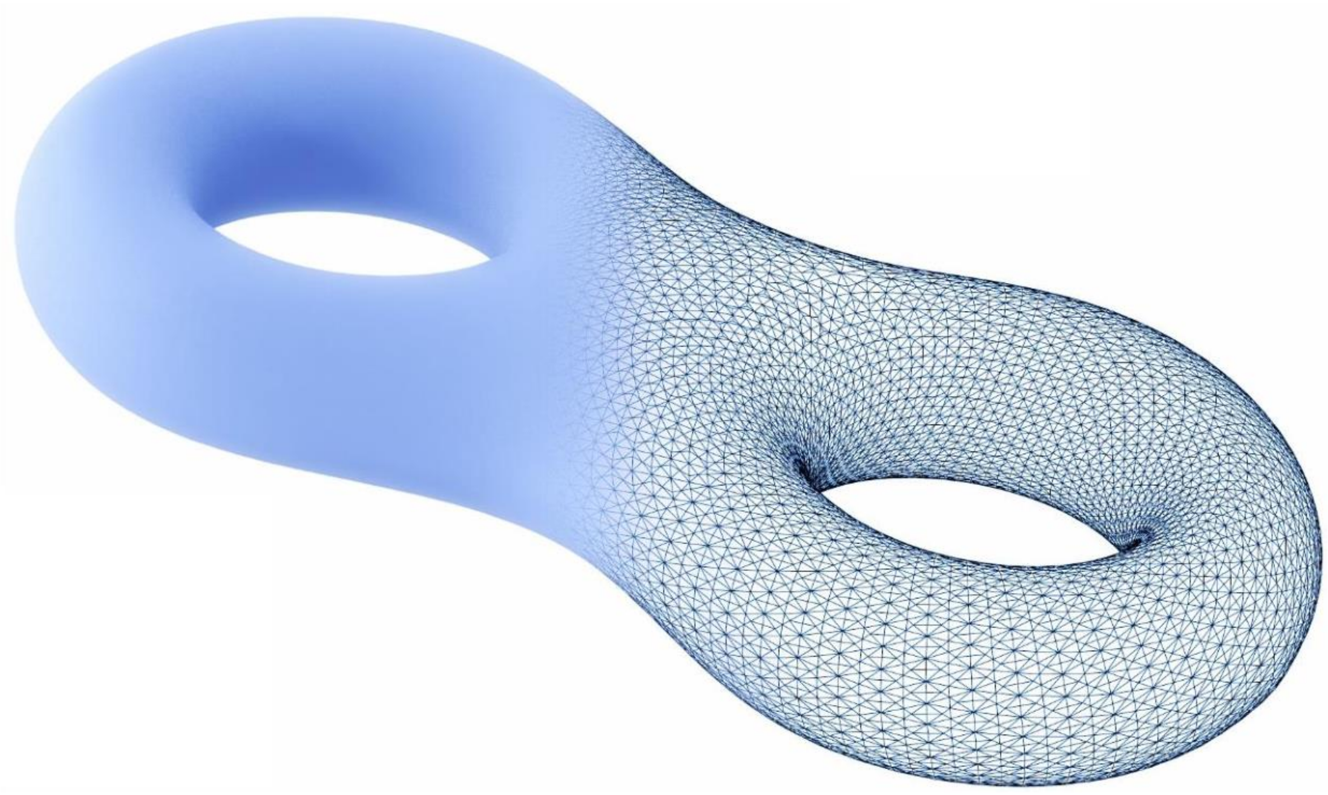
Solve its eigenvalues →

All of them are going to be real and non-negative

# Eigenvalues and Eignenvector of the Laplacian

We can think about a mesh as a graph. We can compute the eigenvalues
And eigenvectors of the Laplacian of this graph



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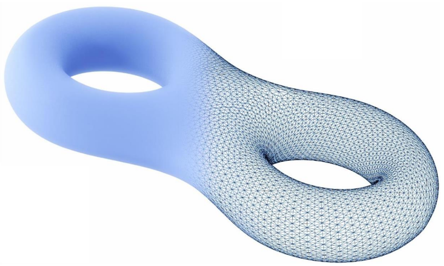
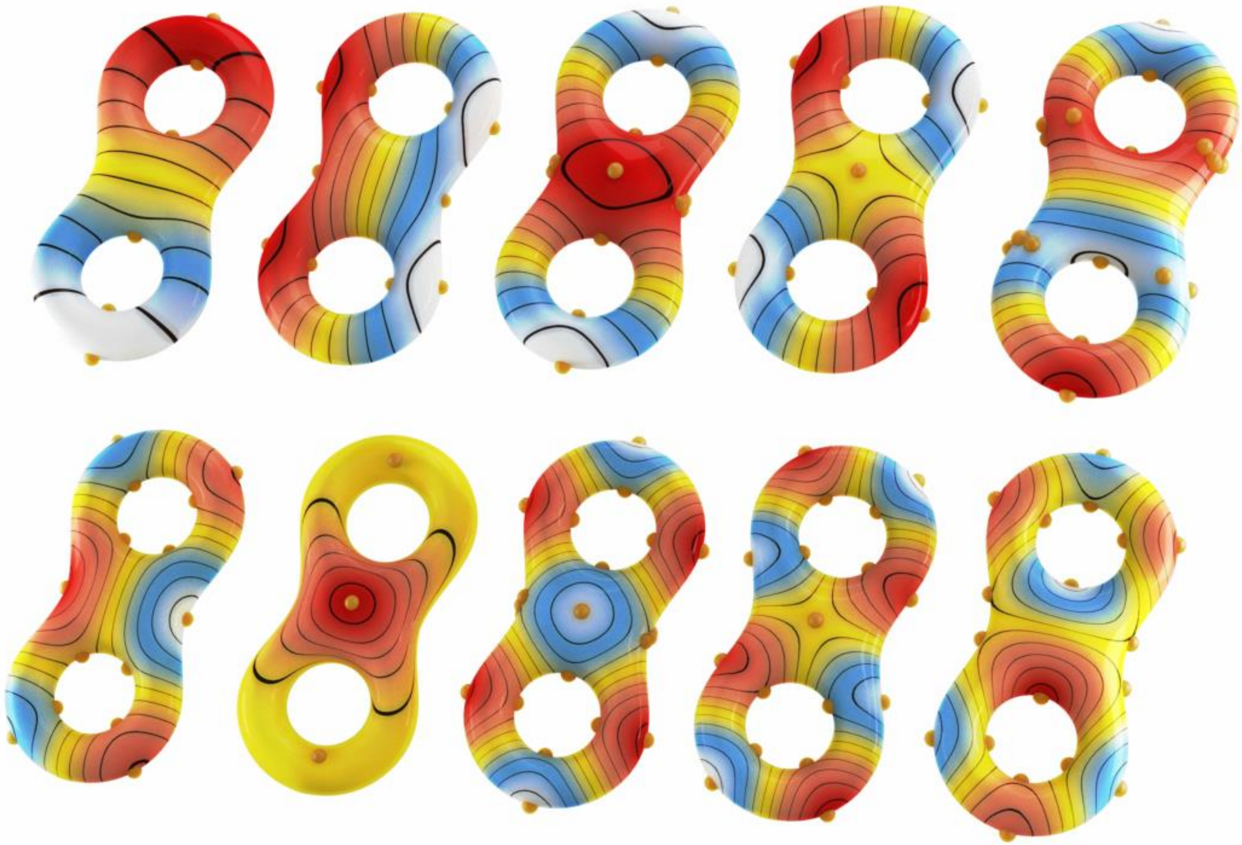




The first 10 eigenvectors of this mesh

# Eigenvalues and Eignenvector in Python

In python you can compute the eigenvalues and the eigenvectors of a matrix : numpy.linalg.eig

# From the data to the graph

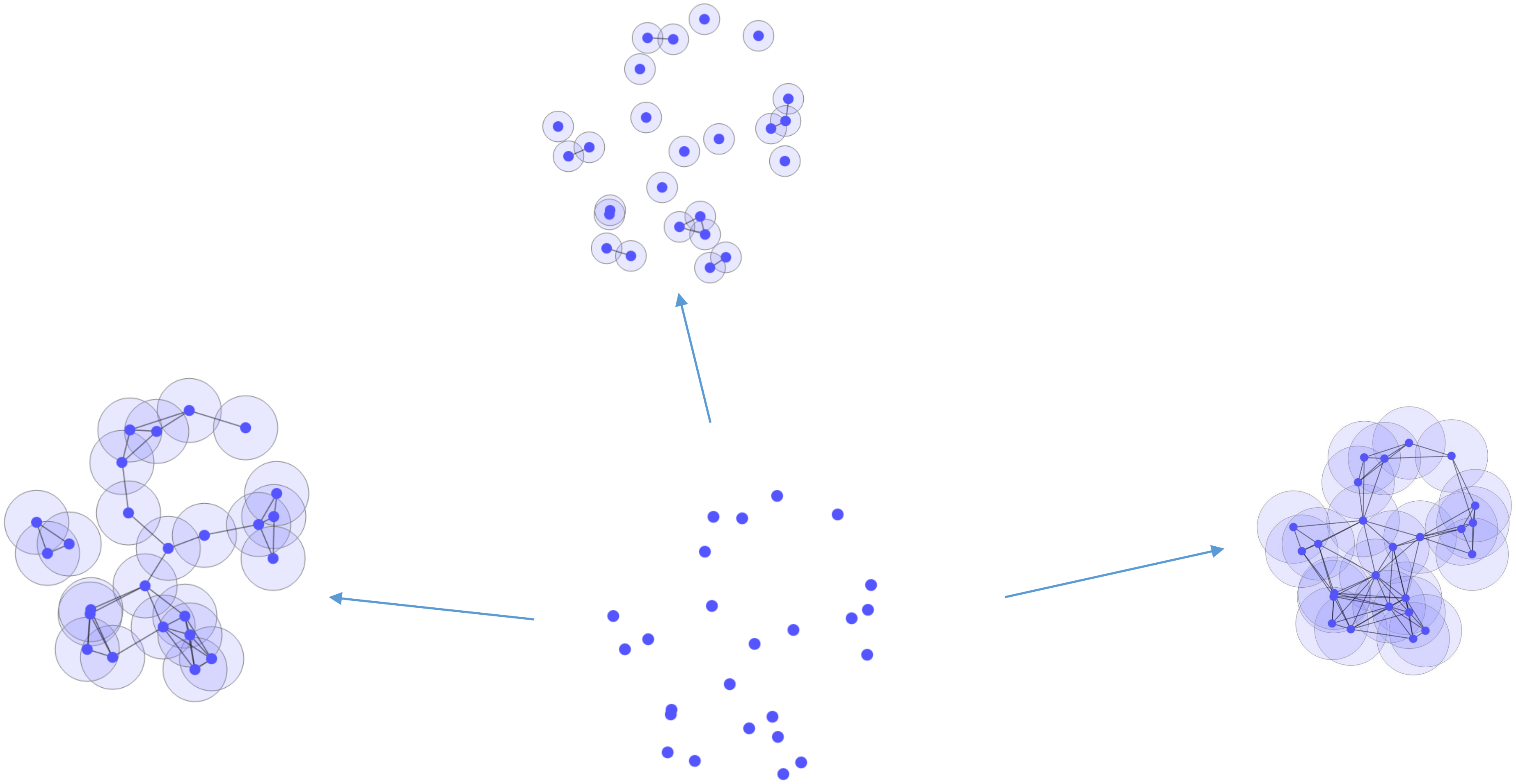
Given the data $X = \{p_1, p_2, \ldots, p_n\}$ , we begin by constructing a graph G on the top of the data X:

- The points in $X$ are the vertices of the graph
- The edges in the graph and their weights are determined by how close together and are in X

Three common methods to construct graphs :

- The neighborhood graphs ($\varepsilon -$ neighborhood graph or the knn graph)
- The complete graph on the set $X$.

# Similarity Graph:ε- Neighborhood Graph



$$Construct\ the\ \varepsilon - neighborhood\ graph$$

A common problem here is which ε we should choose?

# Similarity Graph: The fully connected graph

Suppose that we are given a set of points $X = \{p_1, p_2, \dots, p_n\}$ in $R^d$. Another way to construct a graph on the top of the data X is by connecting all points in X to each other. In this case we weight all edges by $s_{ij} := s(x_i, x_j)$ defined as follows :

$$s(x_i, x_j) = \exp\left(-\frac{\|x_i - x_j\|^2}{2\sigma^2}\right)$$