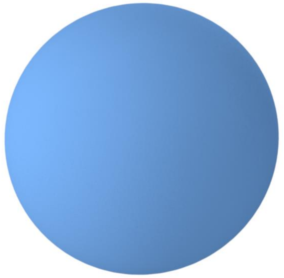
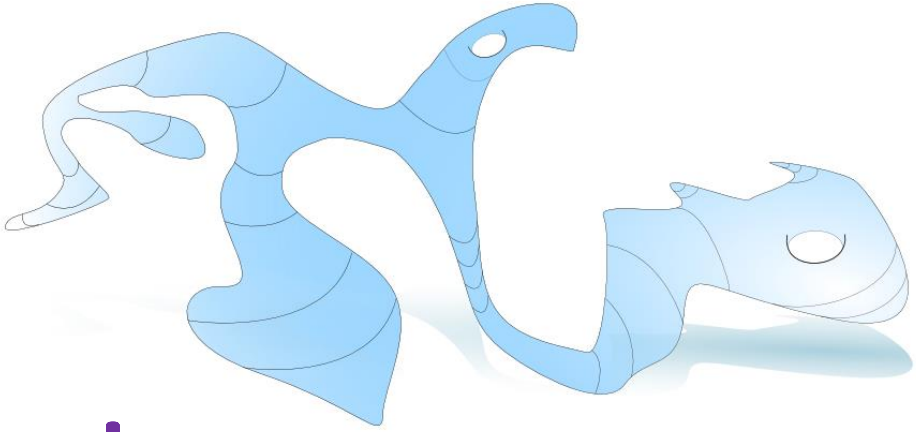
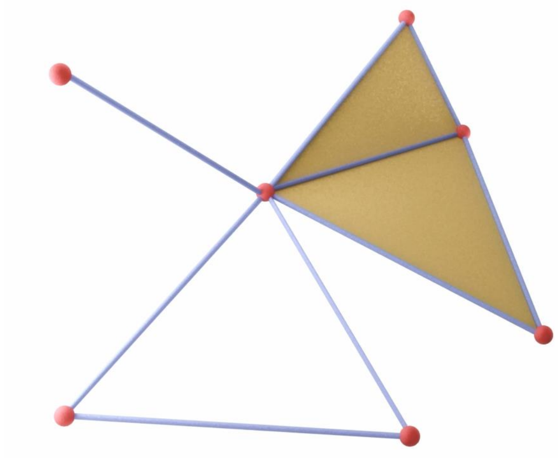




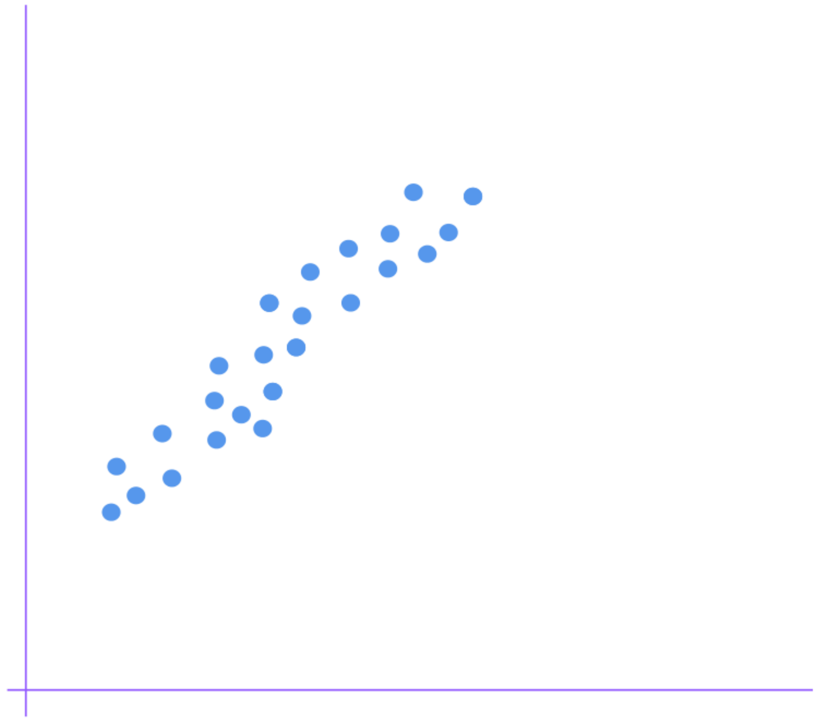
An Introduction to Topological Data Analysis



Mustafa Hajij

Motivation

The classical problem of fitting data set of point in R n using linear regression



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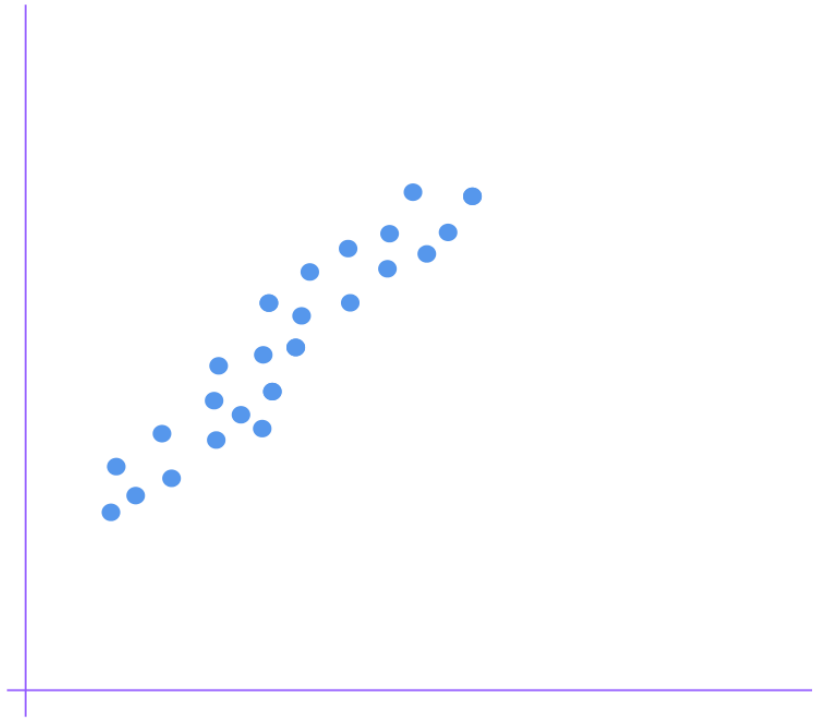

The linear shape of the data is a fundamental assumption underlying the linear regression method

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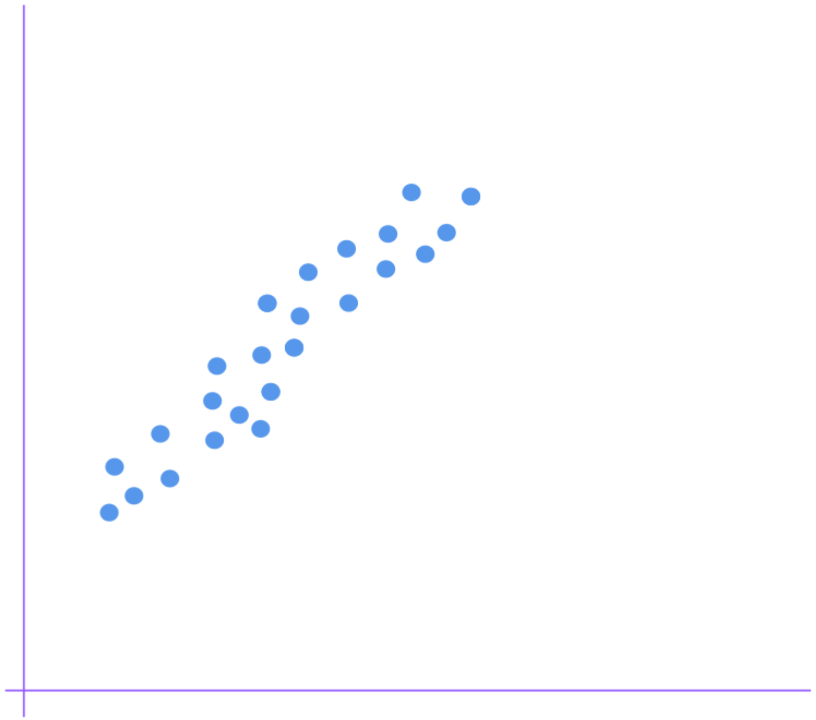

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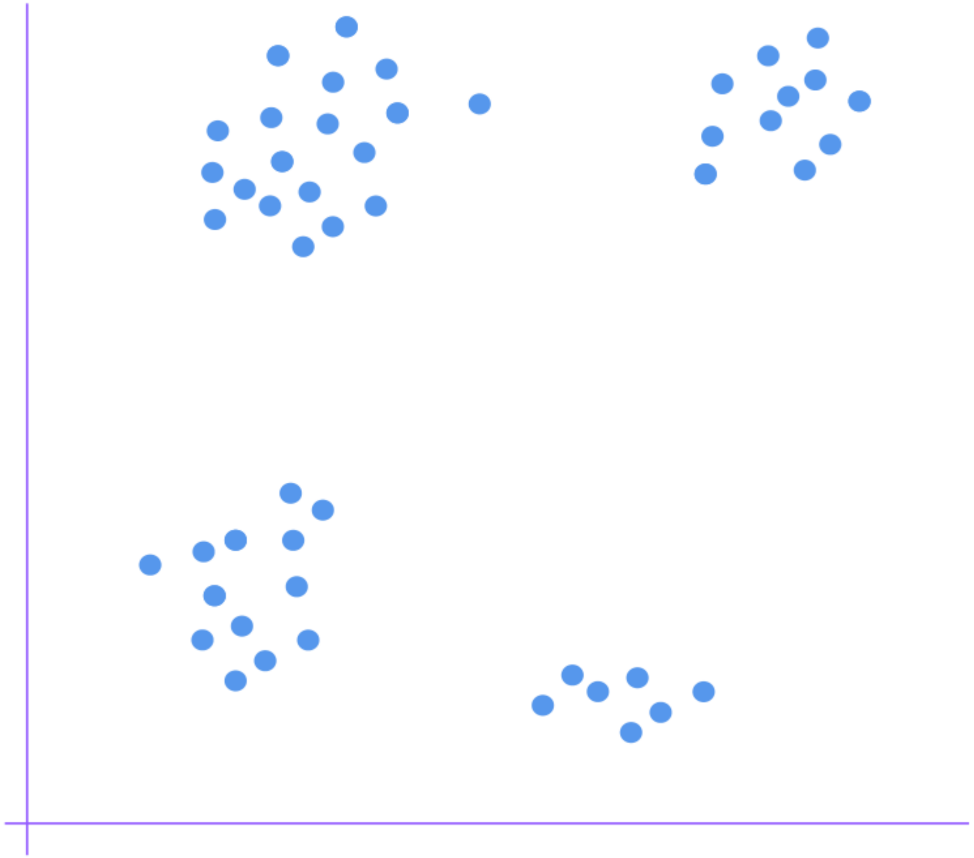
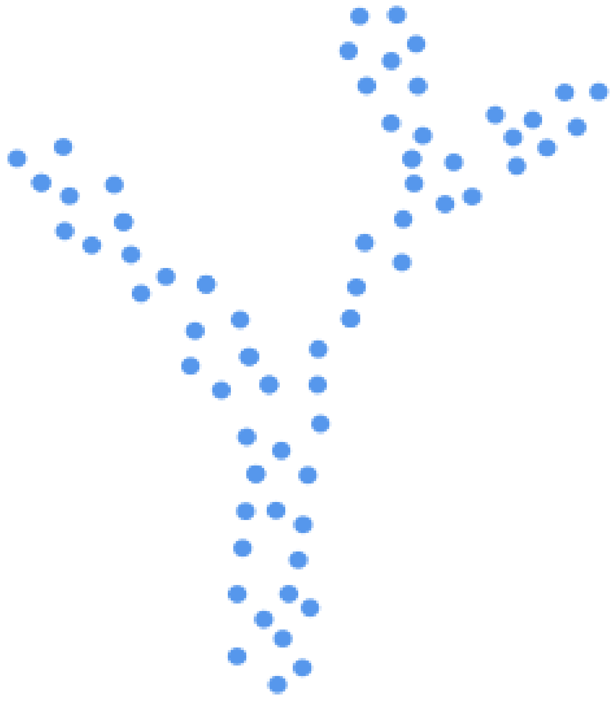
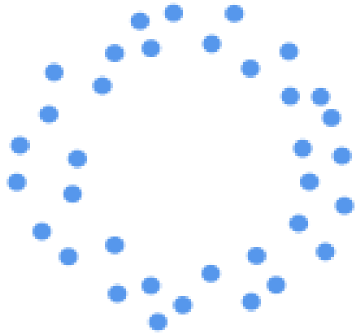


Clustering algorithms assume that the data is clustered in a certain way.

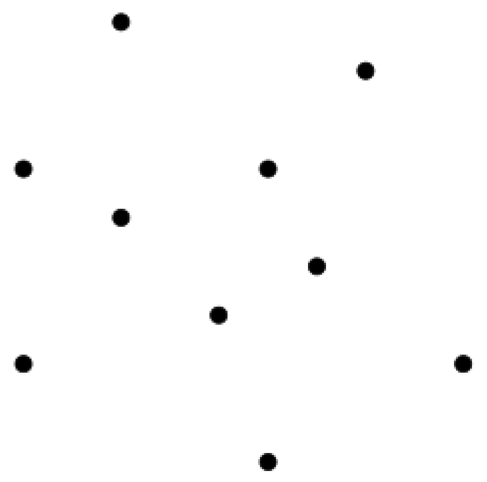
Motivation





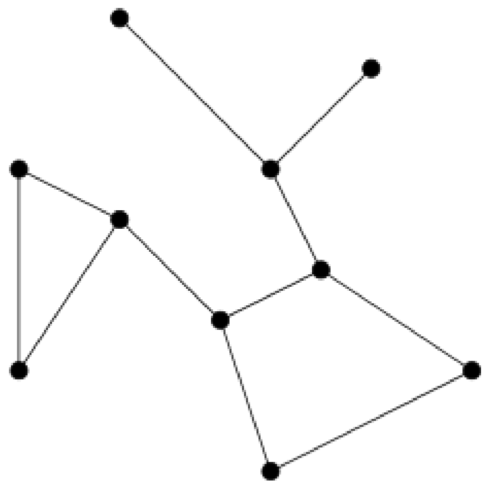
Understanding the shape of the data is a fundamental assumption underlying the analytical method

Concept



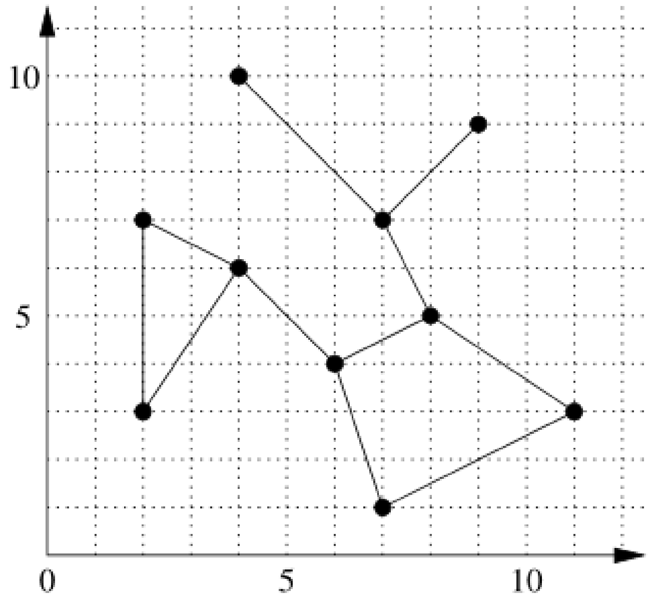
Space

Concept



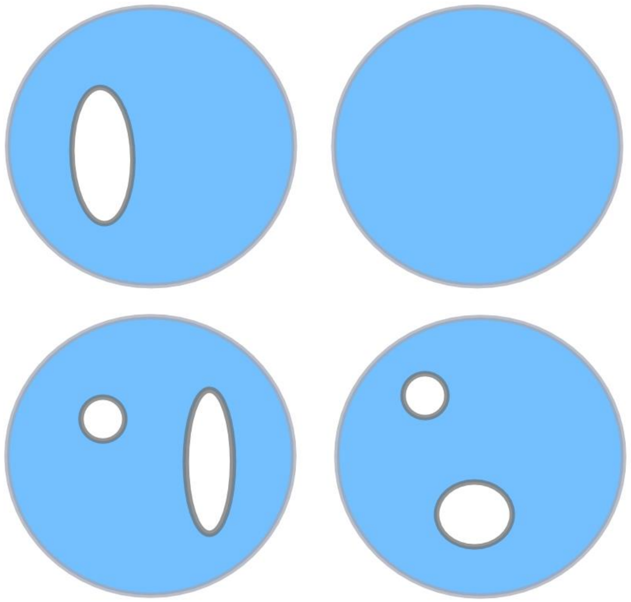
Topological space

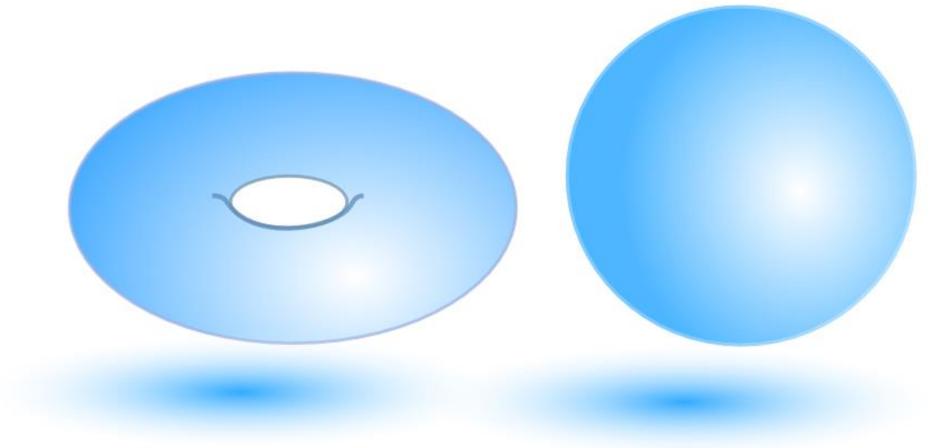
Concept



Metric Space

Concept





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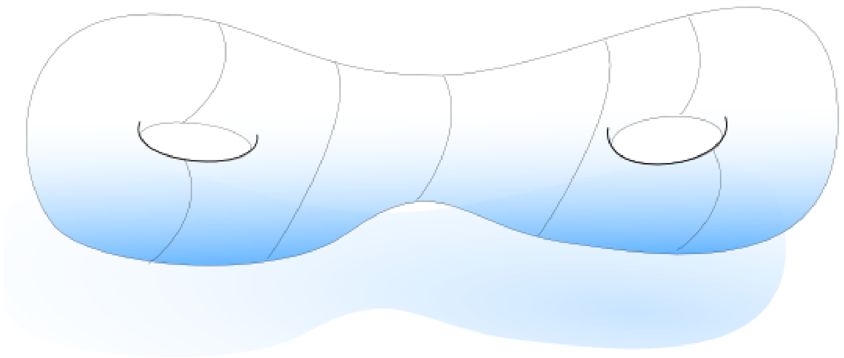

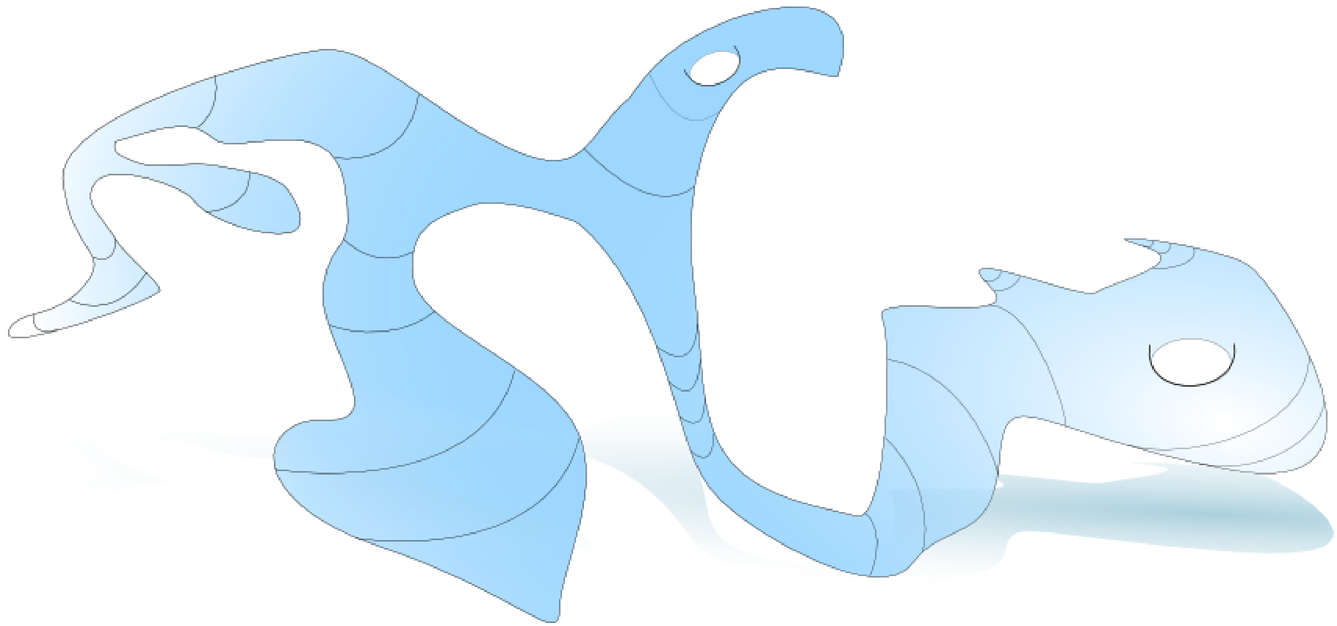

Motivation

Roughly speaking, topology of an object studies the way this object is connected.

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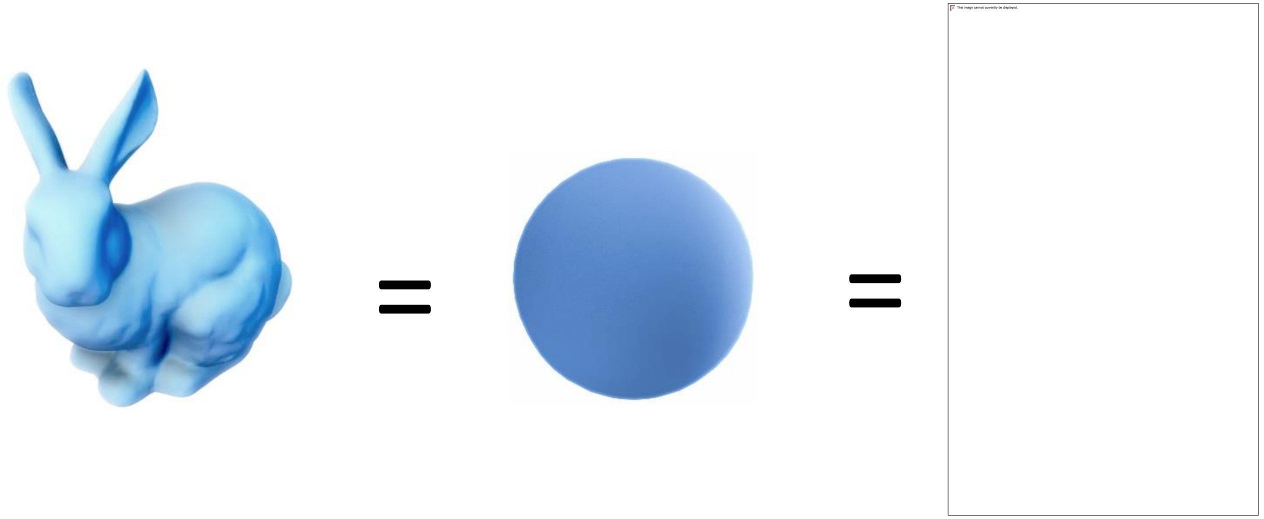

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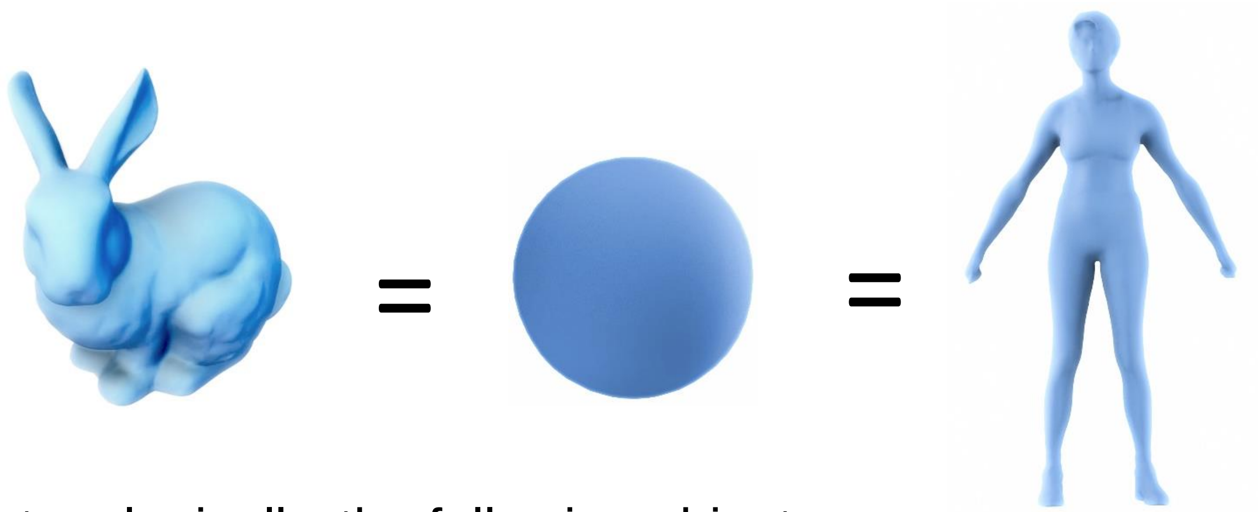

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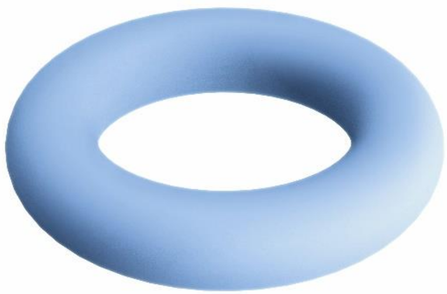

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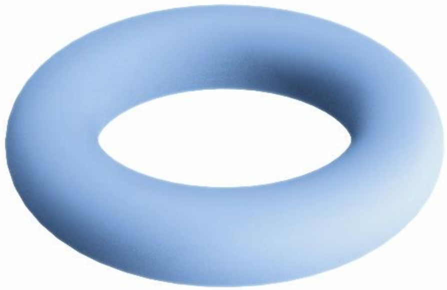

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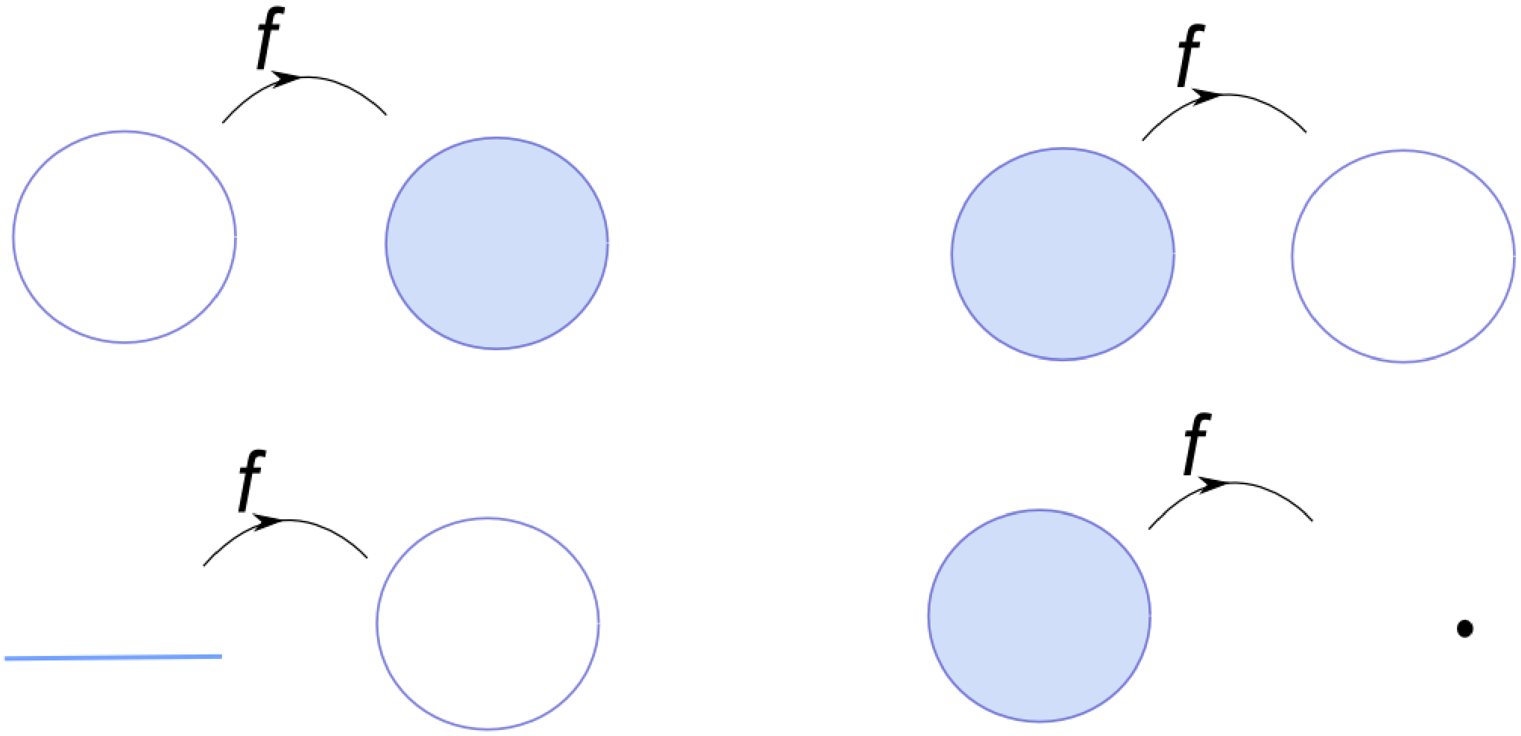
Topology makes these notions precise.

Continuous functions

Which of the following functions is continuous ?



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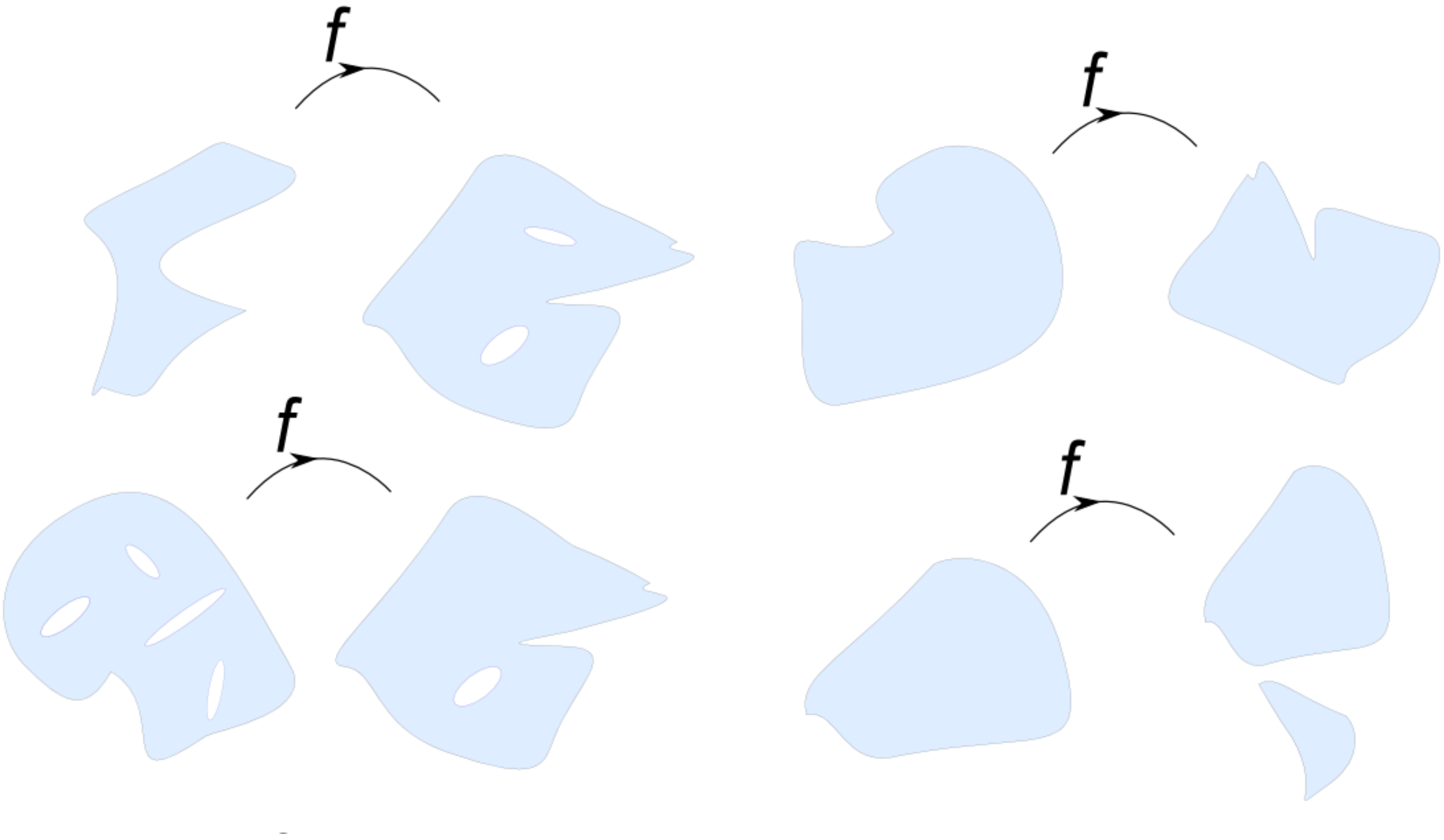
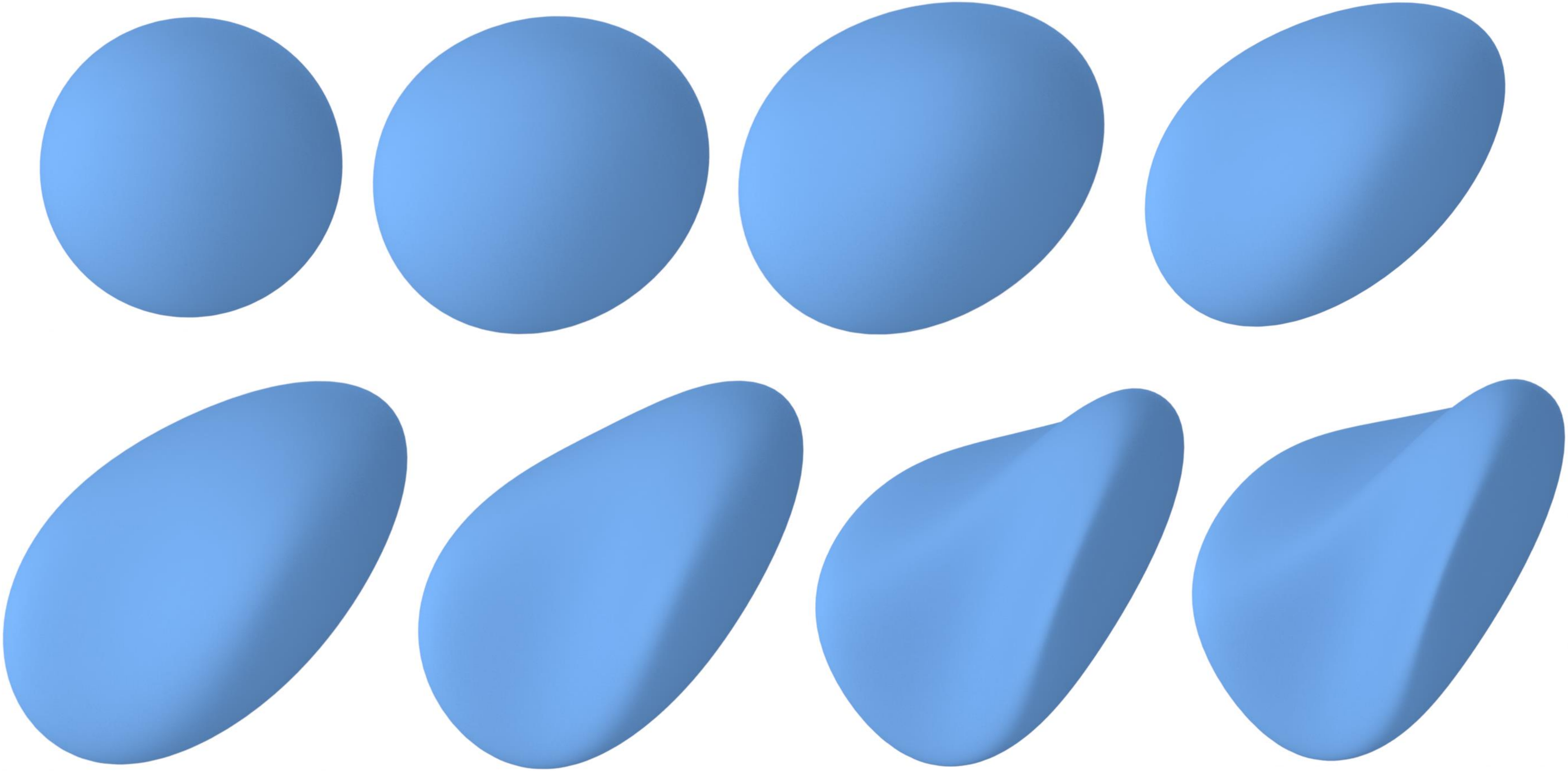


Homeomorphism



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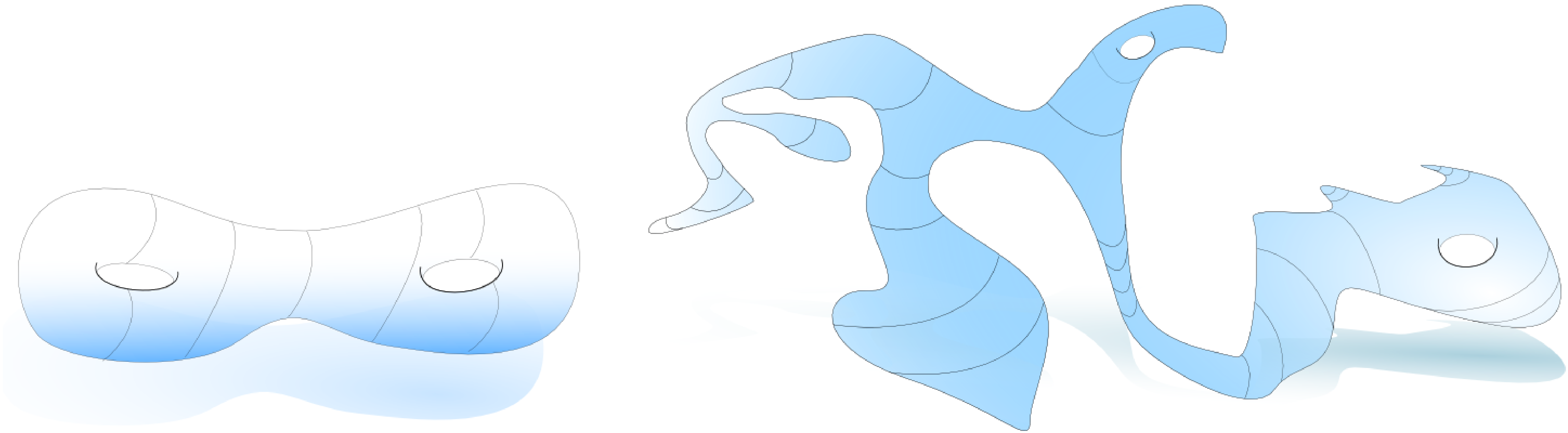
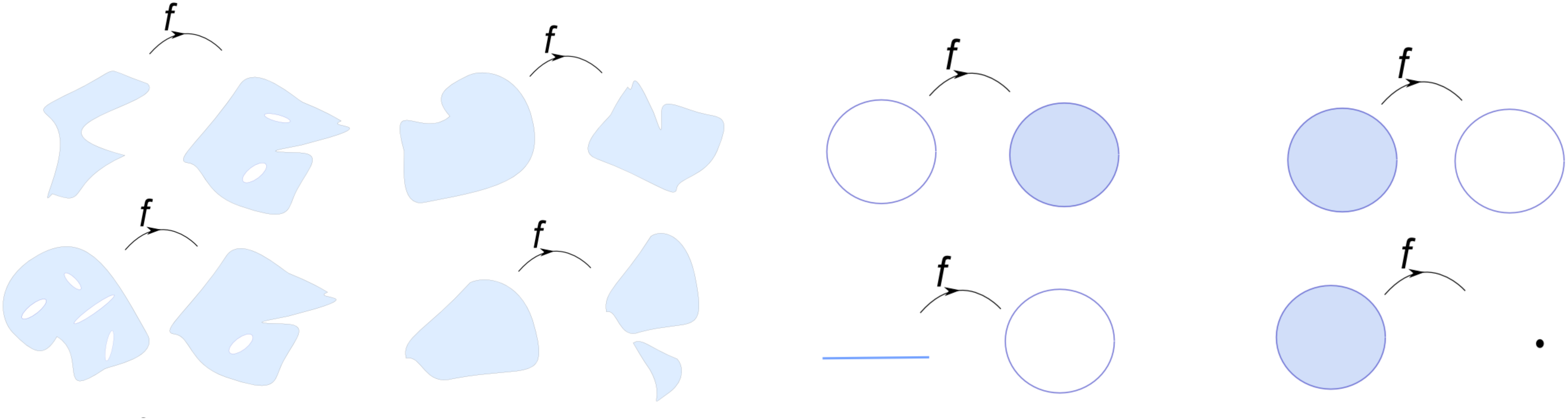
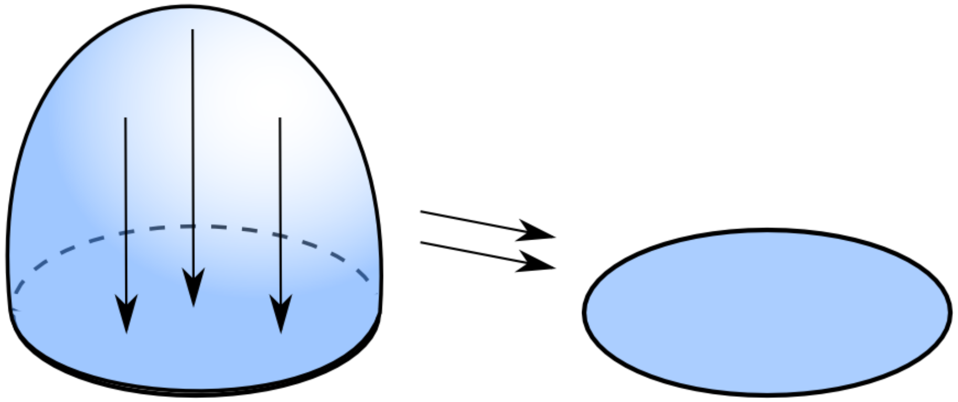


Homeomorphism

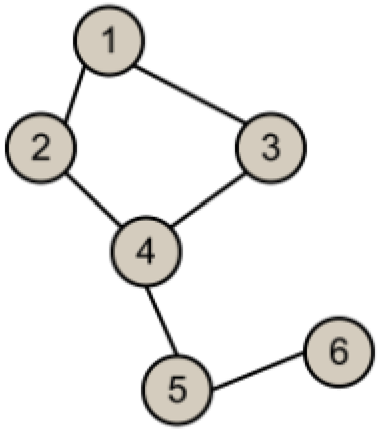
Which of the following functions is homeomorphism? If not, which one of the three conditions is violated?



Homeomorphism-examples



How can we explain a topological space to a computer ?



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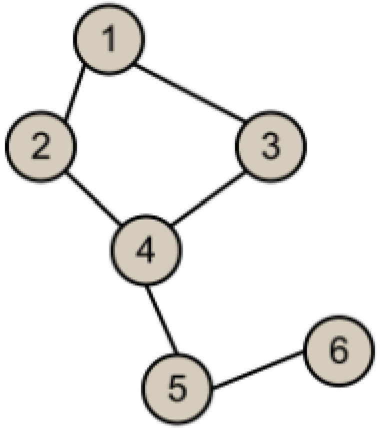

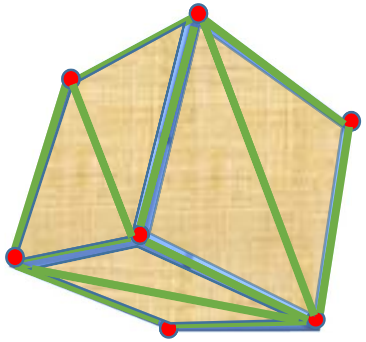

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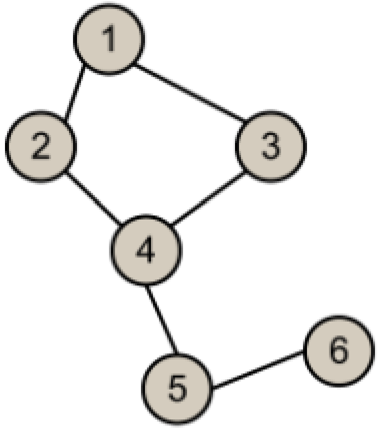
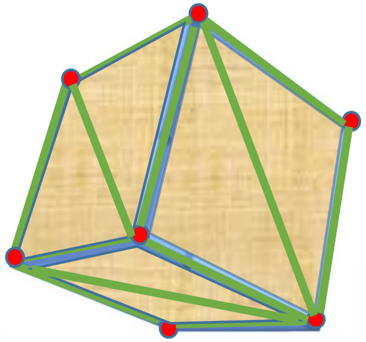
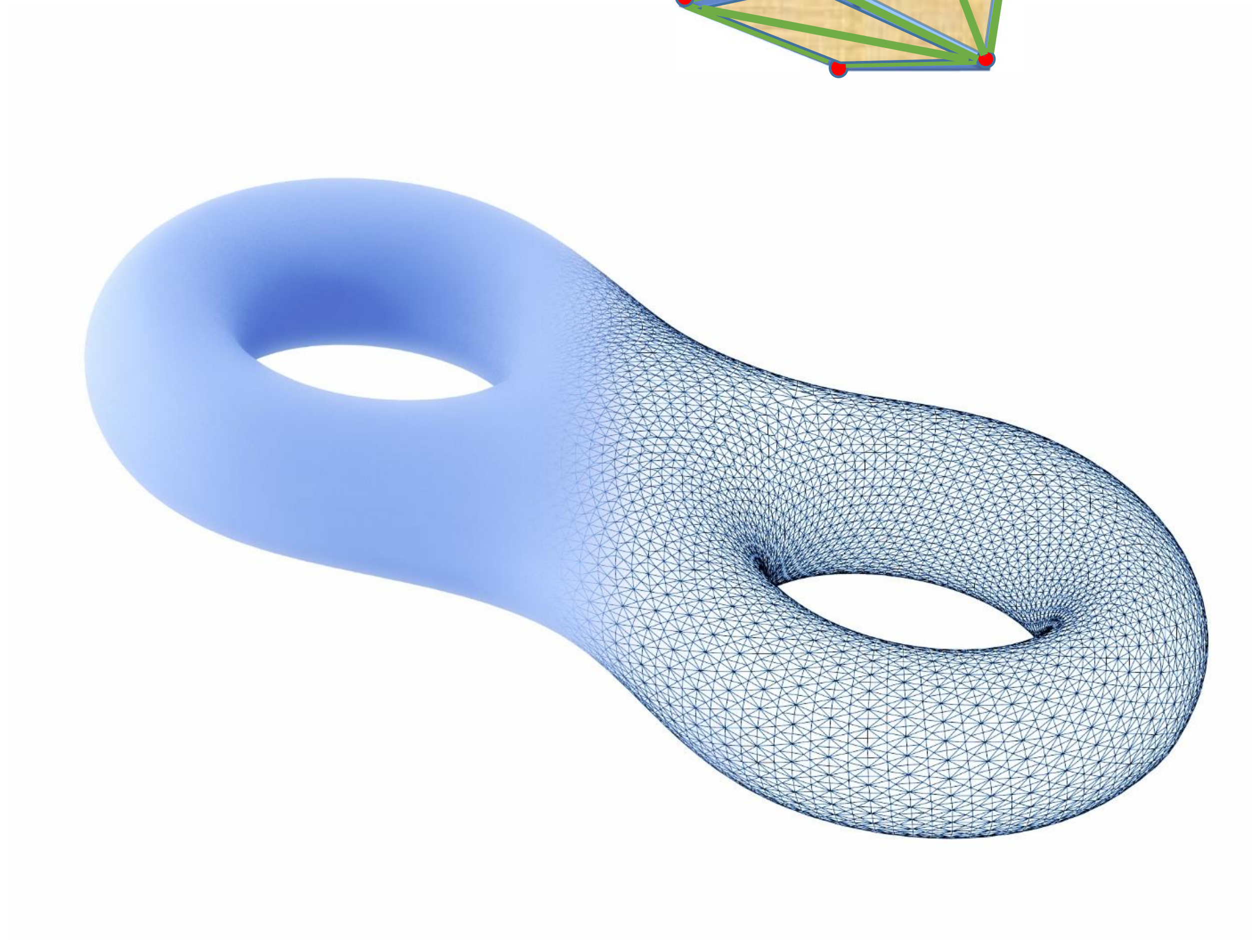
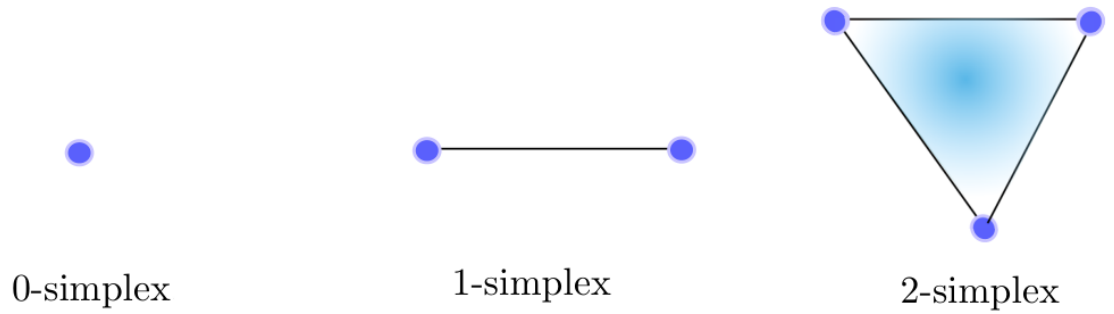






How can we explain a topological space to a computer ?

Key Idea : we use simple building blocks (called simplices)



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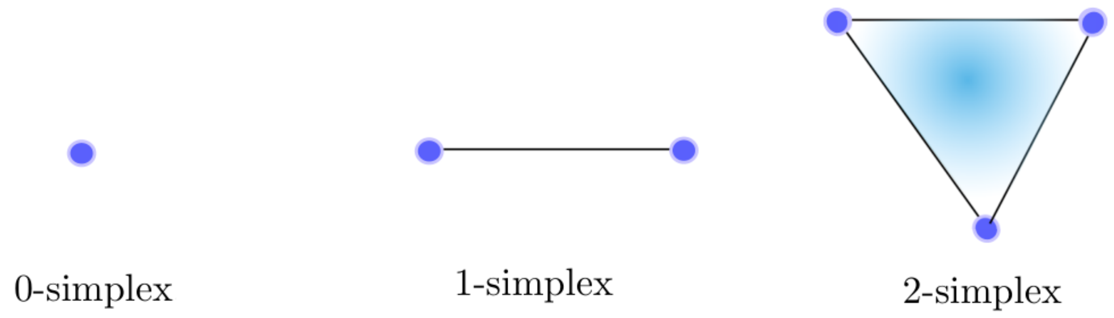
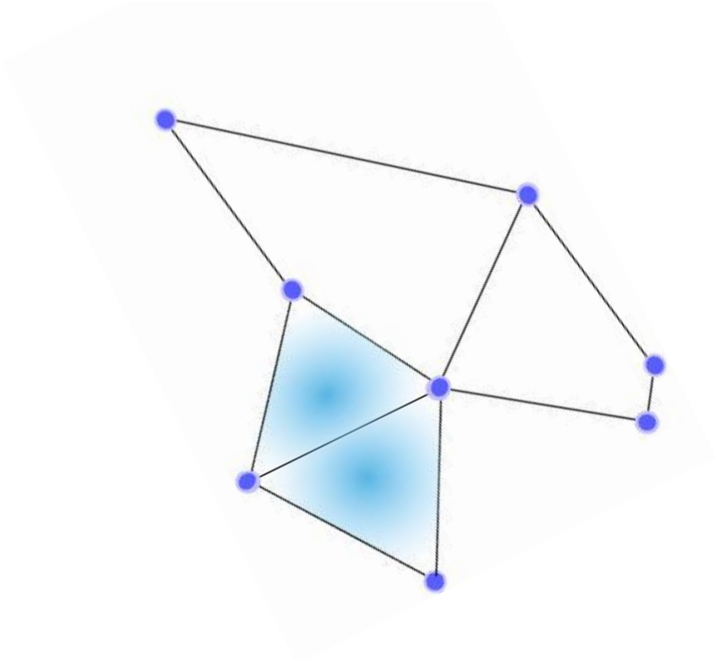
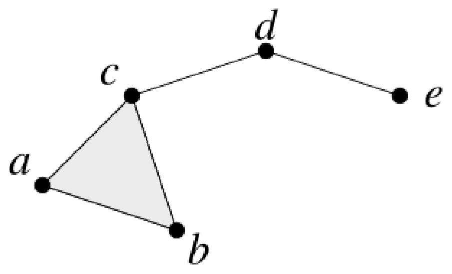


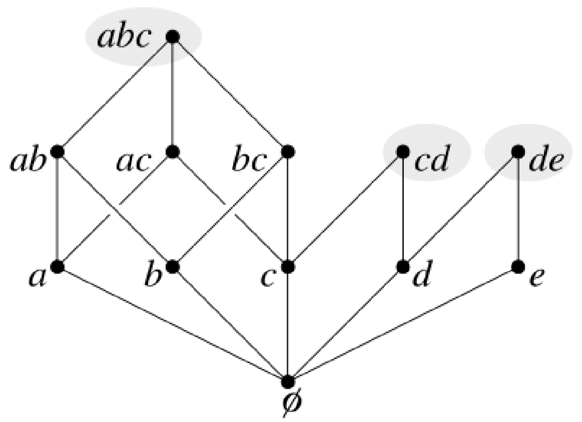


to build more complicated shape.

How can we explain a topological space to a computer ?

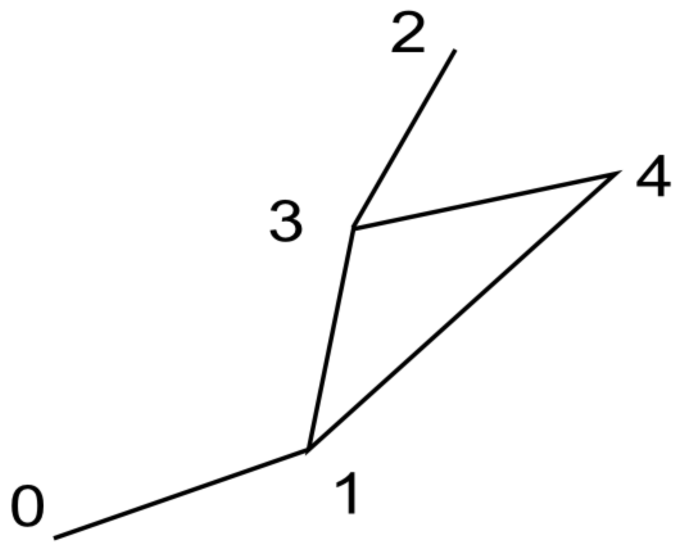


(a) A small complex



(b) Poset of the small complex, with principal simplices marked

Graphs representation: list of edges



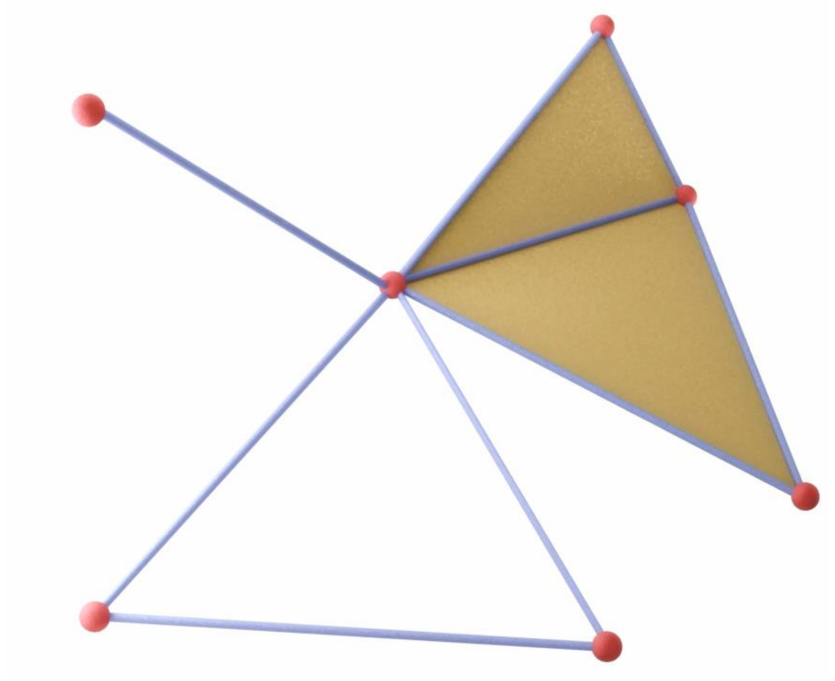
[[0,1], [1,3],[1,4] [3,4], [3,2]]

- *The vertices can be recovered from the edges.*
- *The order of the vertices is important only if the graph is directed.*

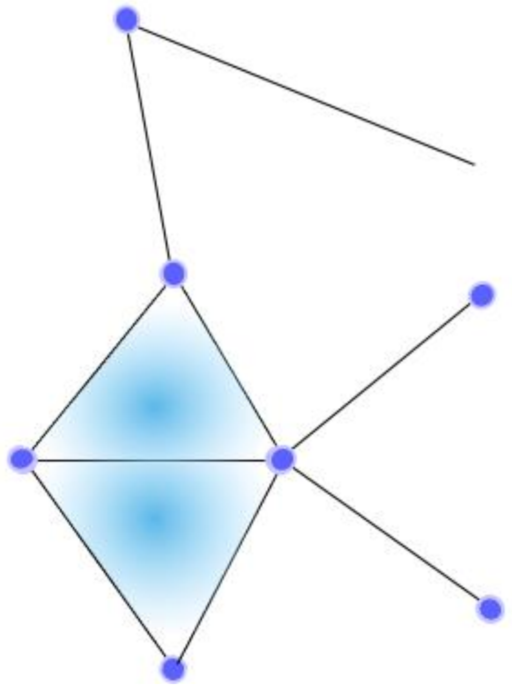
Simplicial Complex-precise definition

Definition A simplical complex is a finite collection of simplices Σ that satisfies the following conditions:

(1) If σ is in Σ then all the facets of σ are also in Σ.
(2) If $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2 \neq \phi$, then $\sigma_1 \cap \sigma_2$ is in Σ.



Examples and non-examples



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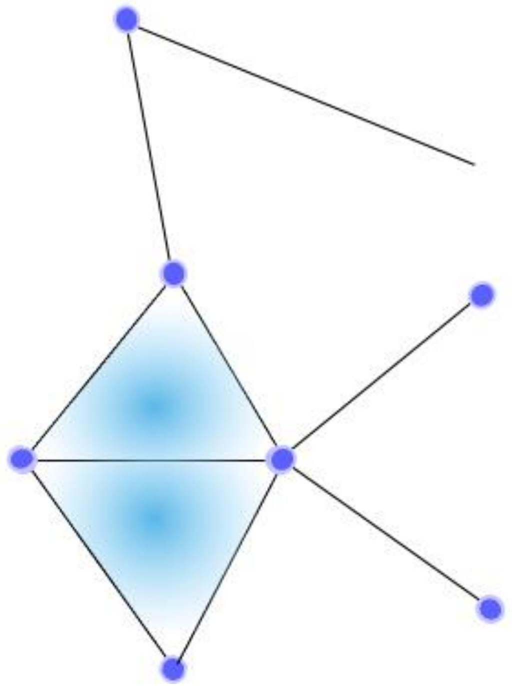

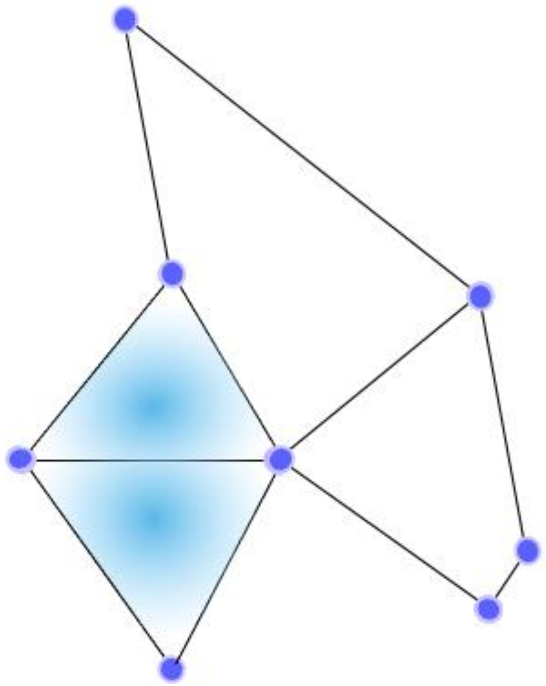

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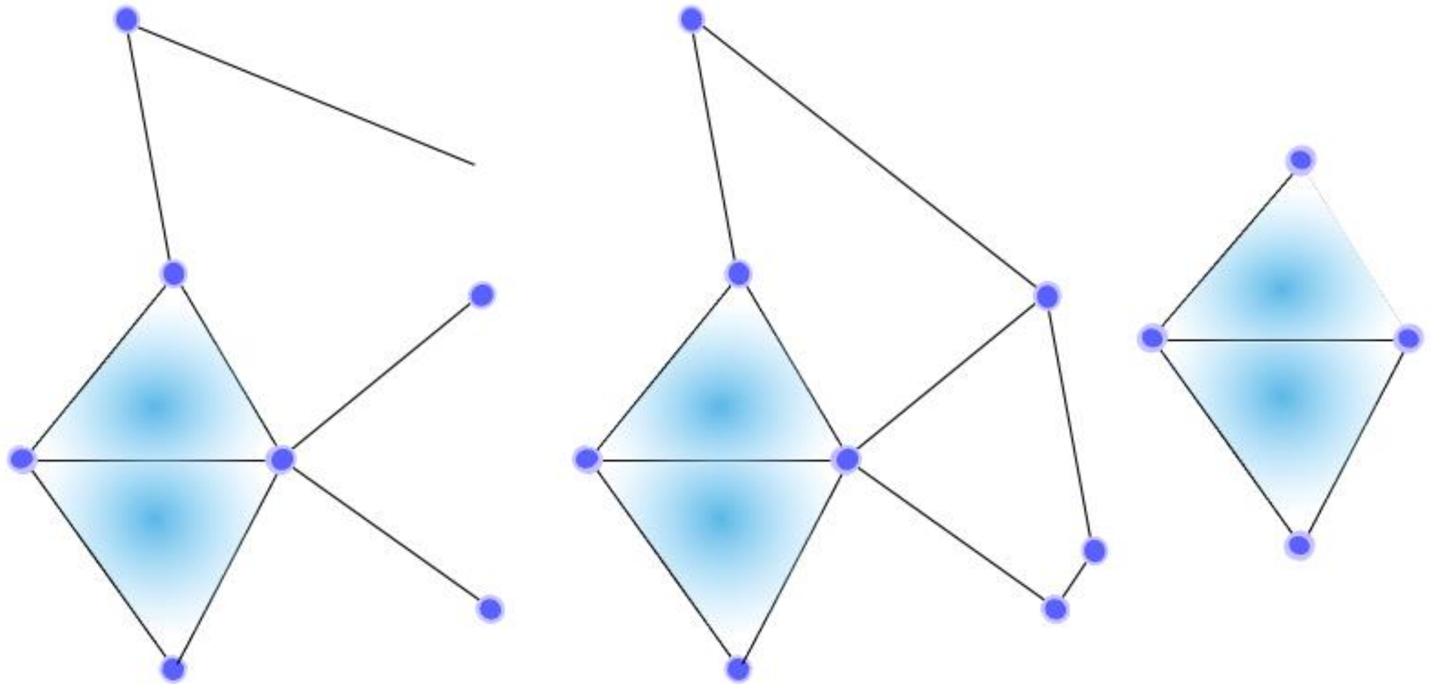

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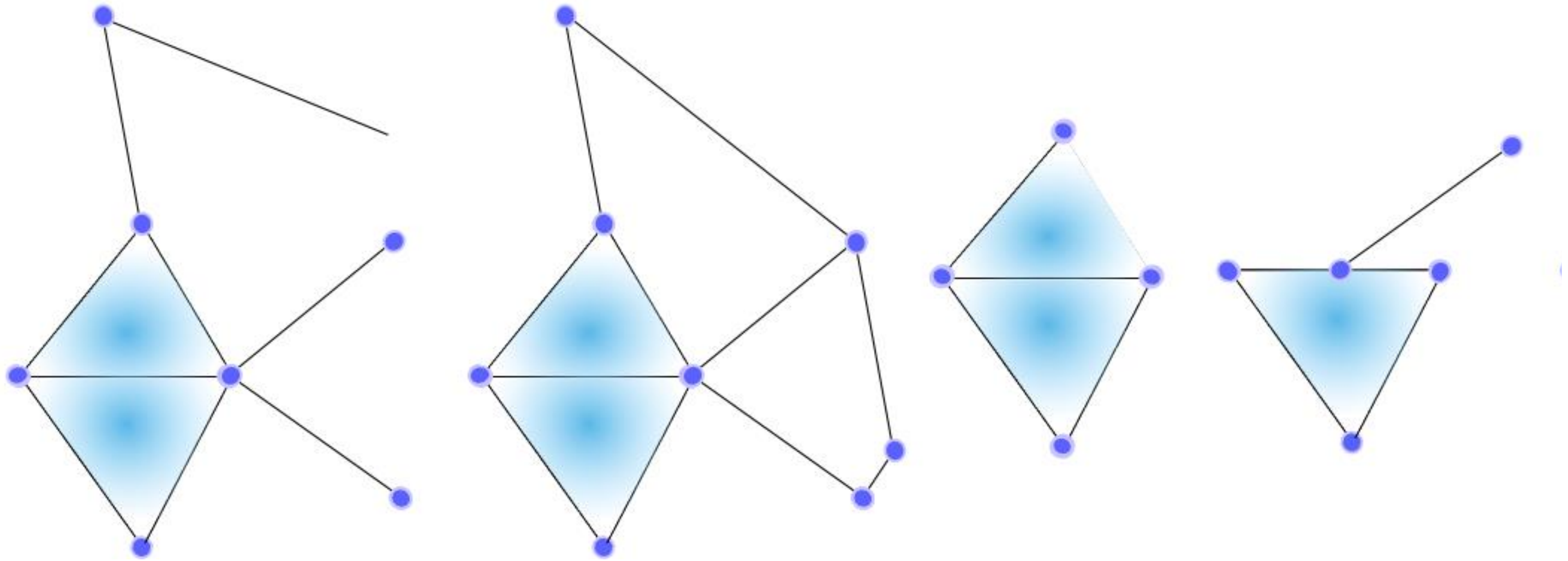

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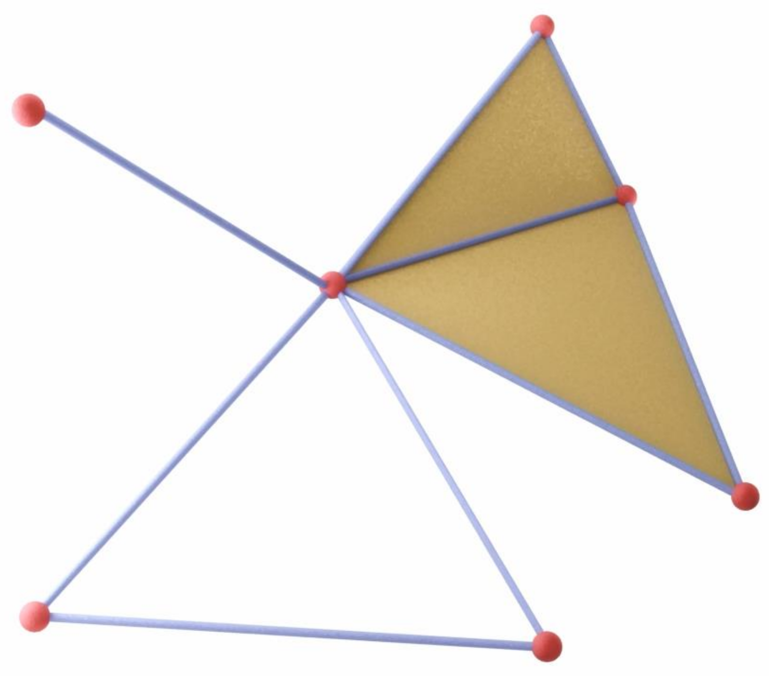
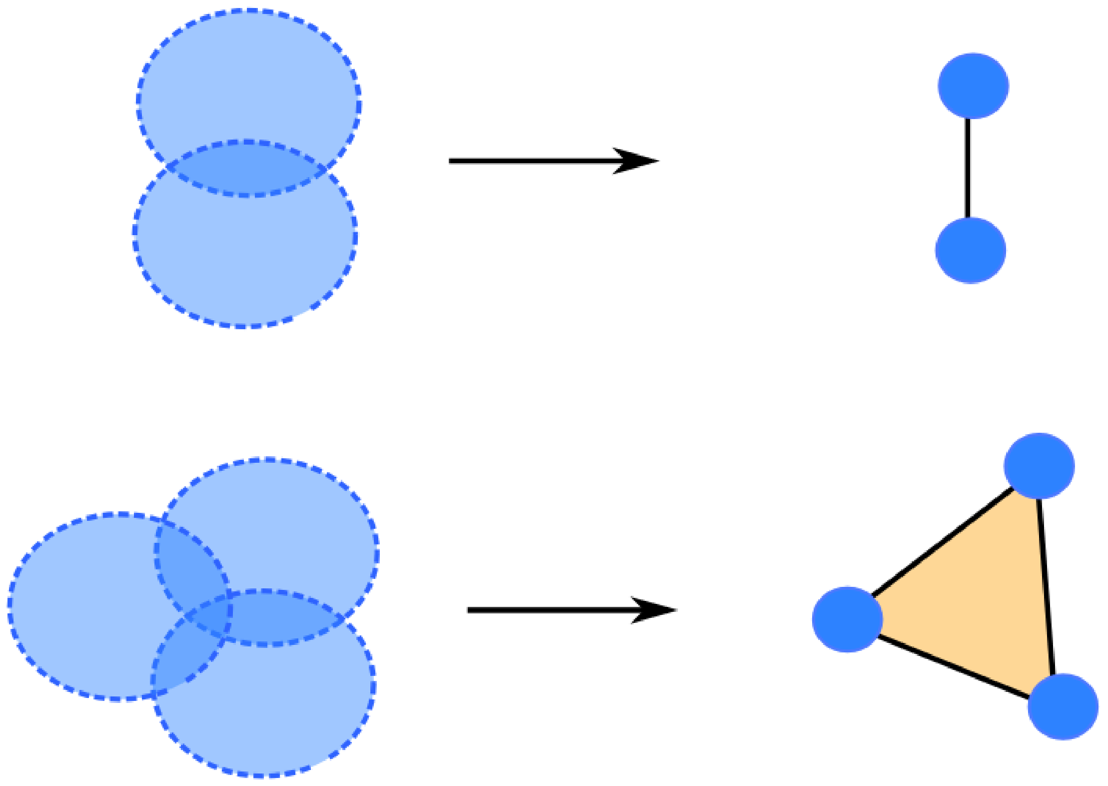
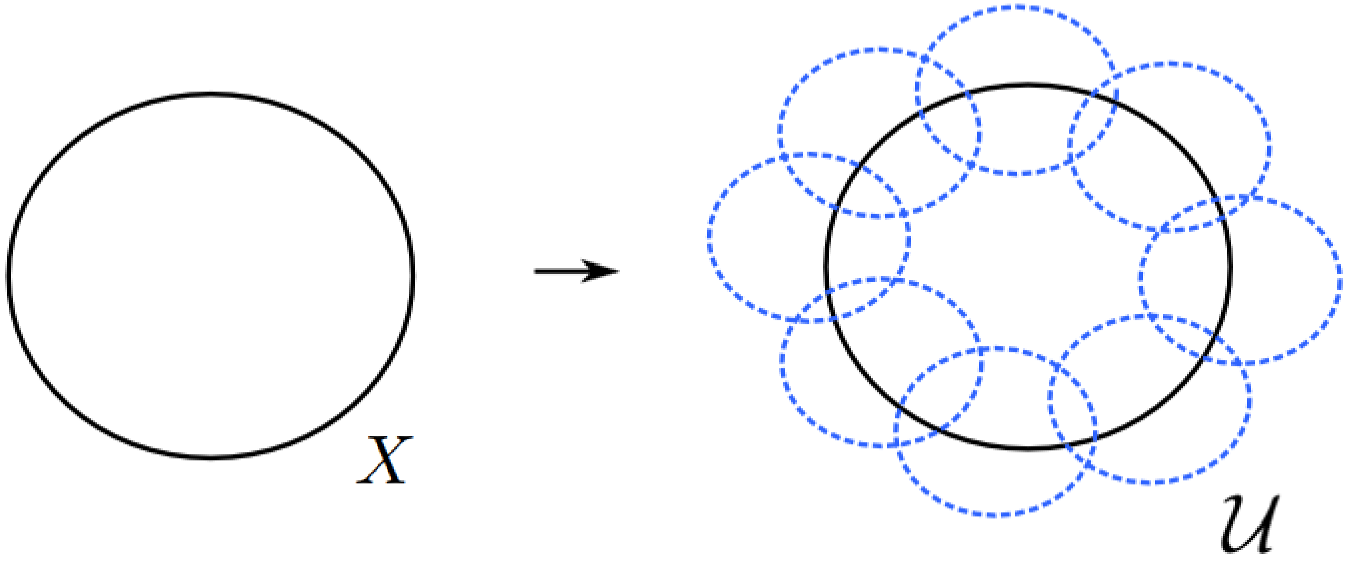


Approximation of the shape



Cover of a space

A cover of a space X is a collection of sets U whose union is the entire space



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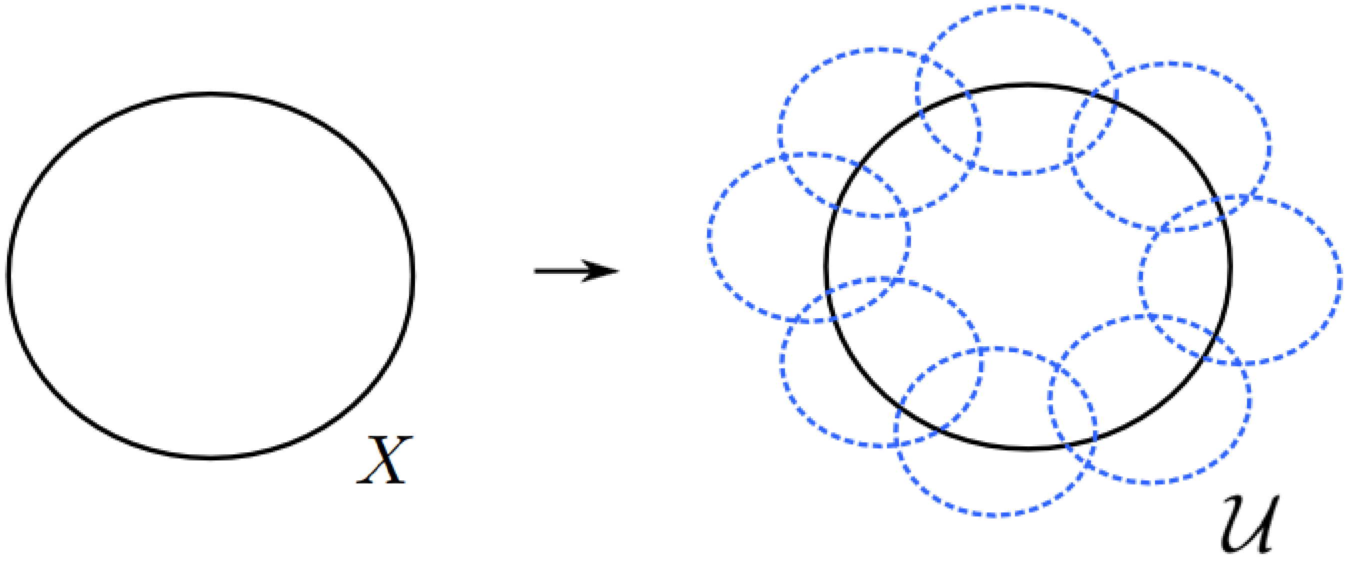
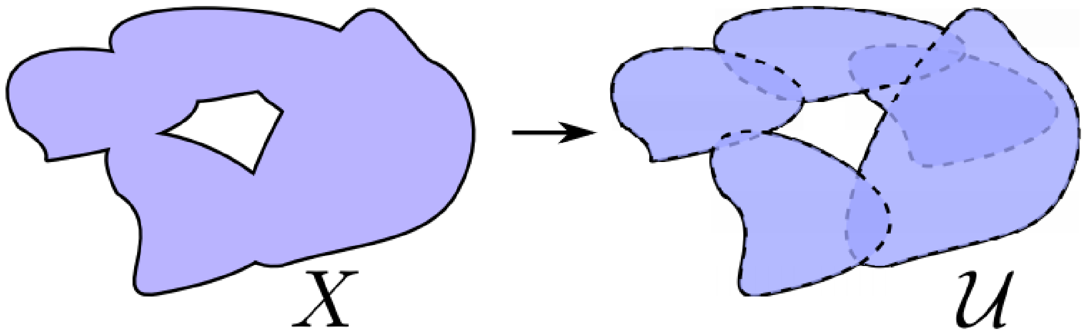
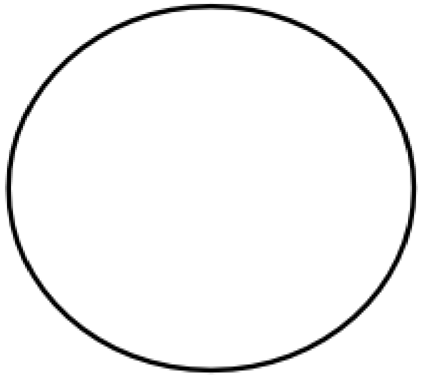




Approximation of the shape

Covers can be used to obtain an approximation for the underlying data



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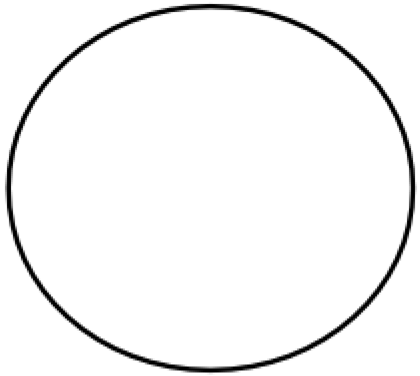

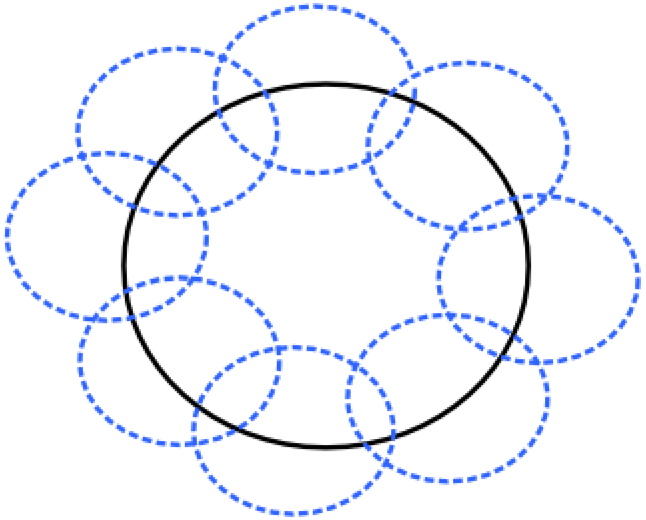

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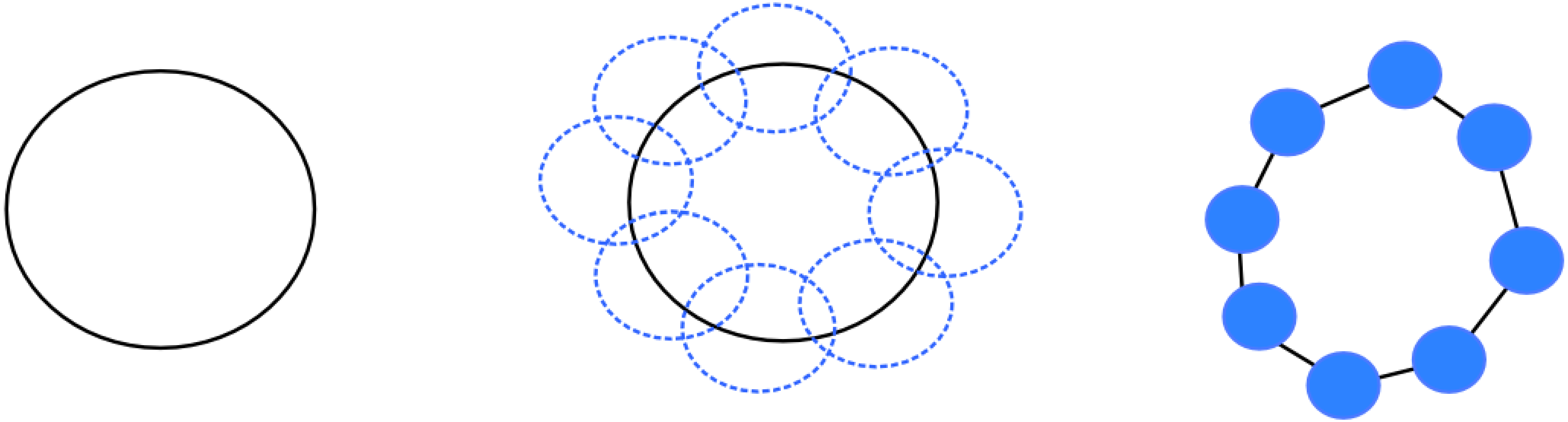
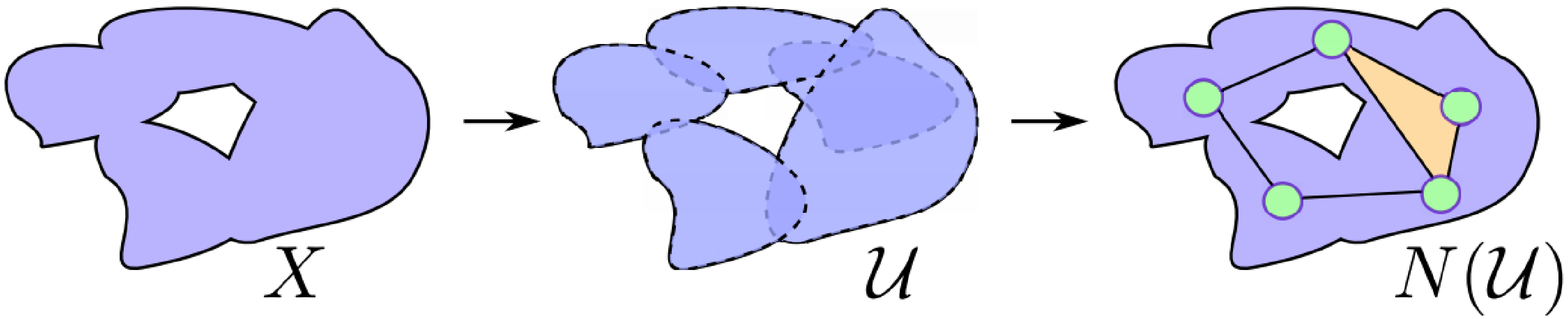


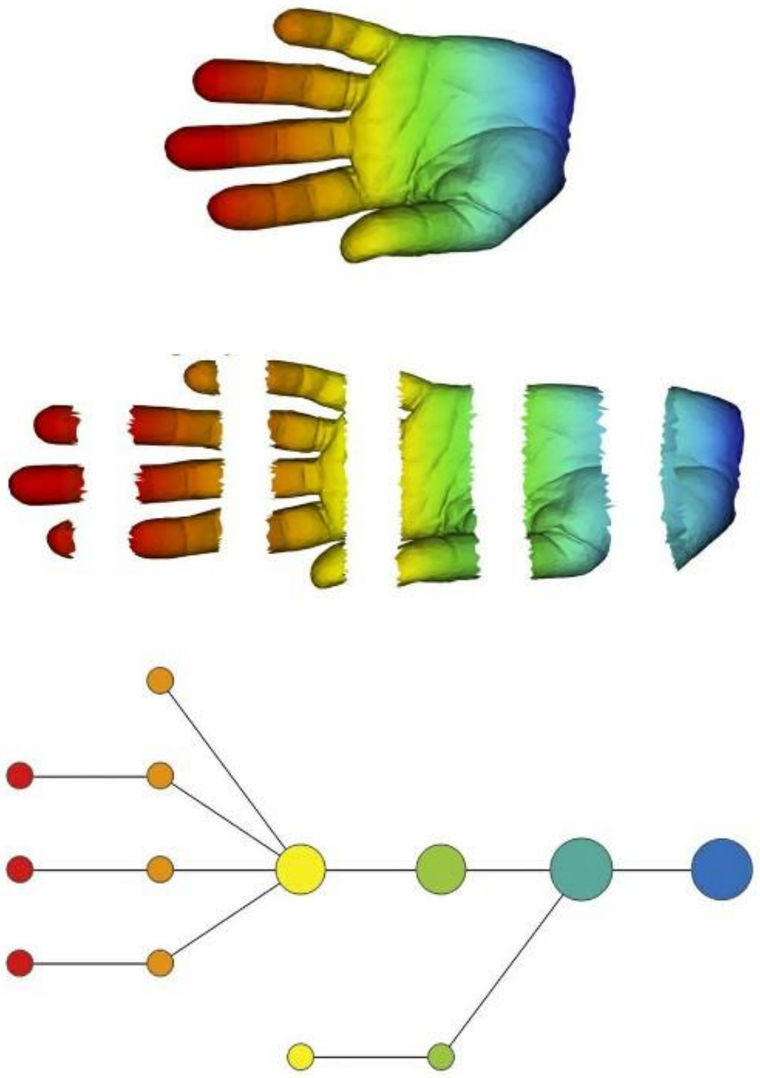
Nerve of a space

Approximation of the shape



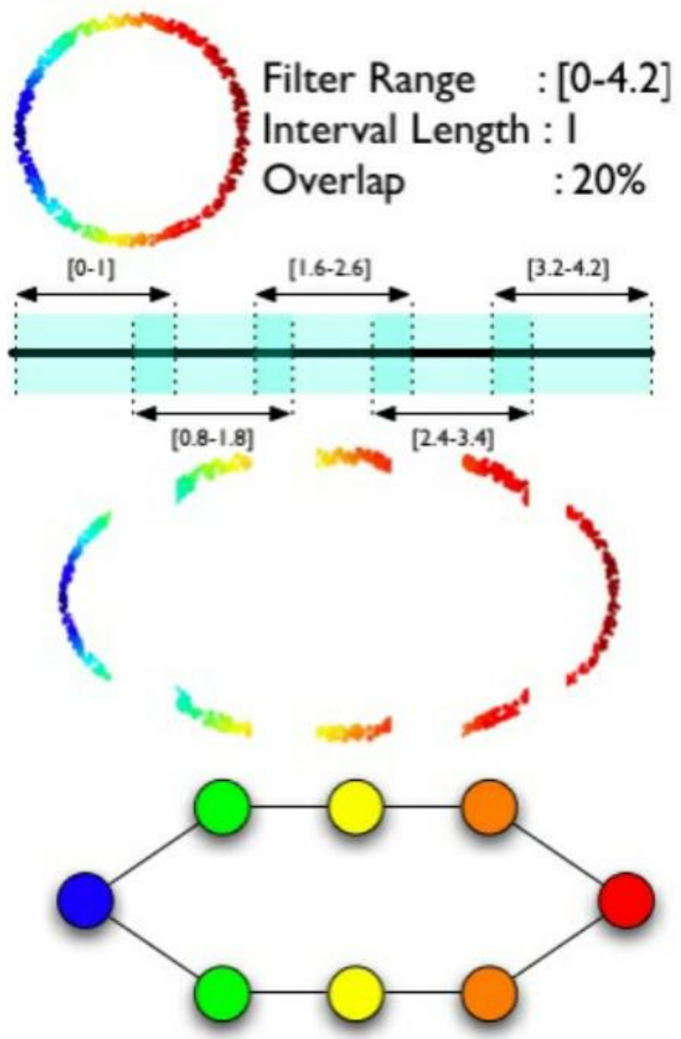
Key idea: every set is replaced by a node
every intersection is replaced by an edge
if we have intersection between three sets we replace them with a face and so on.

Mapper Construction



source

Mapper Construction



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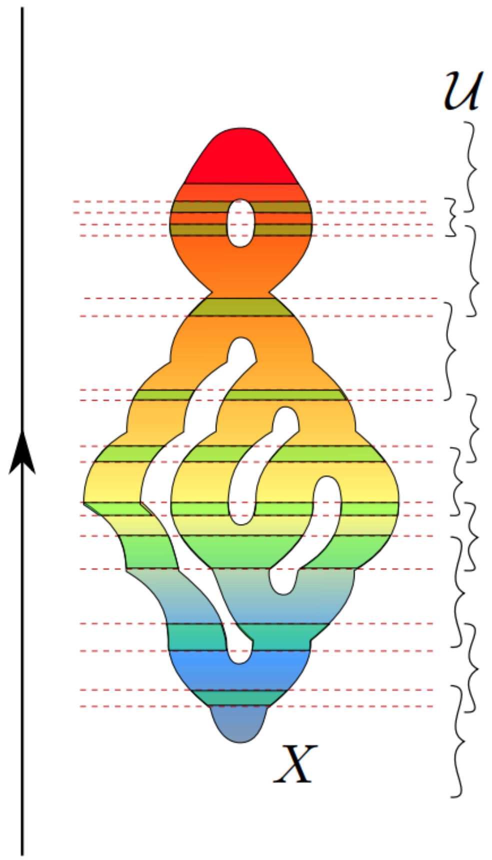

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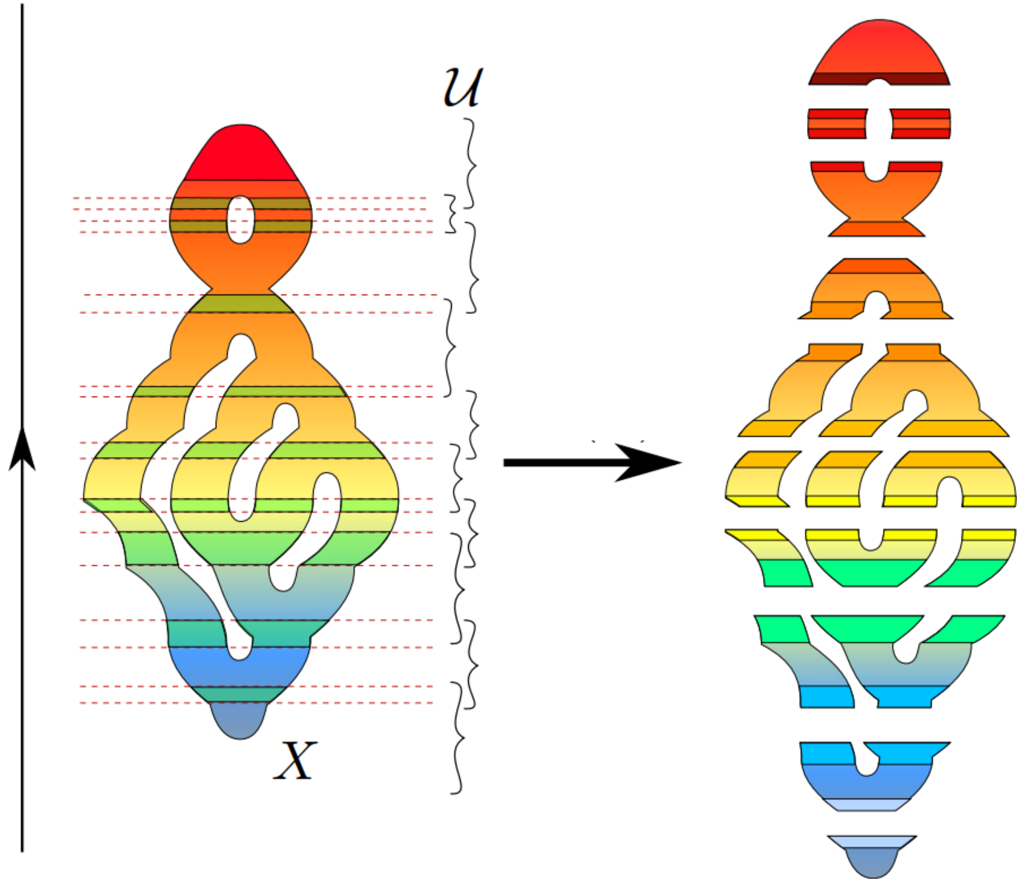

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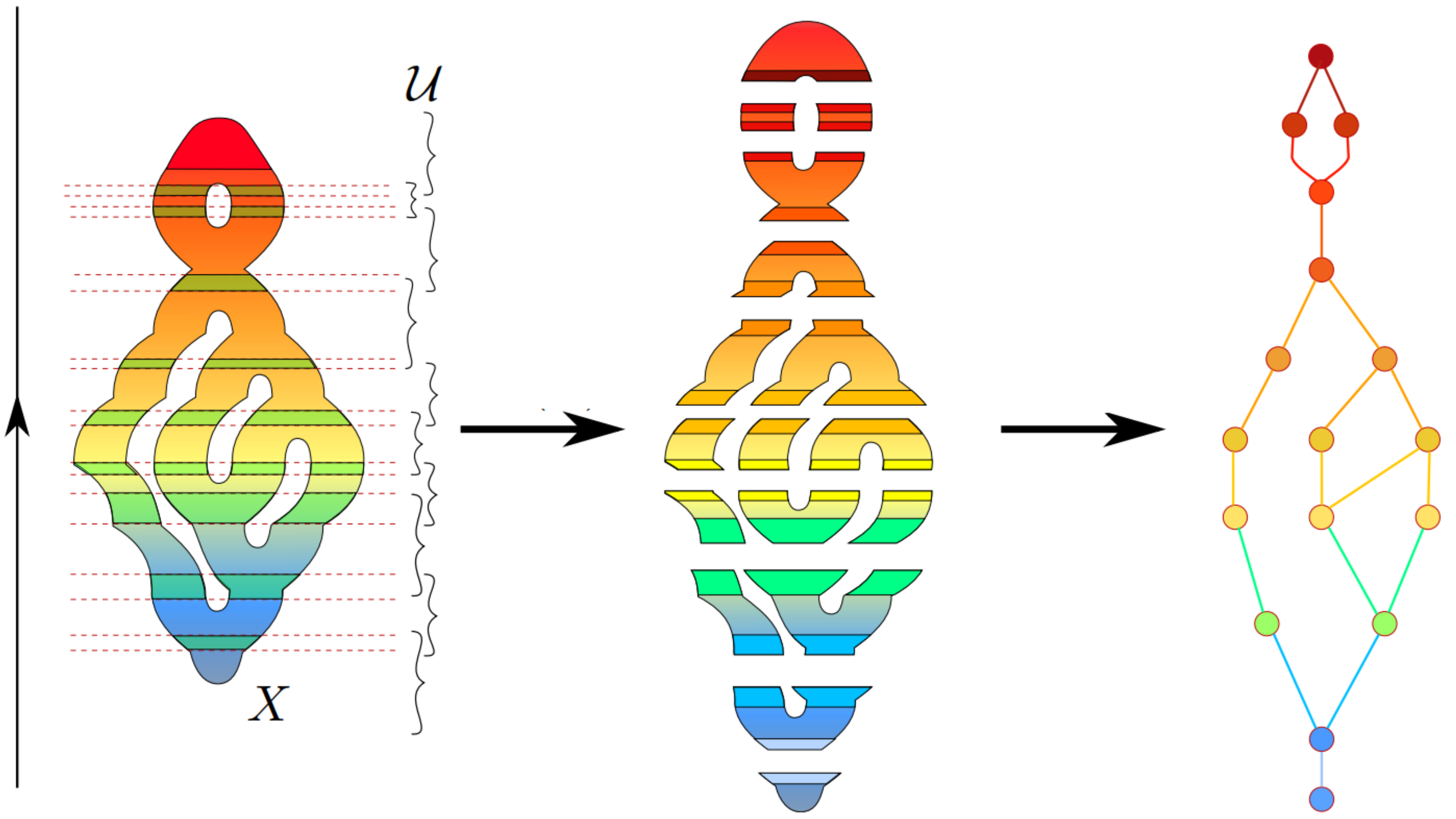
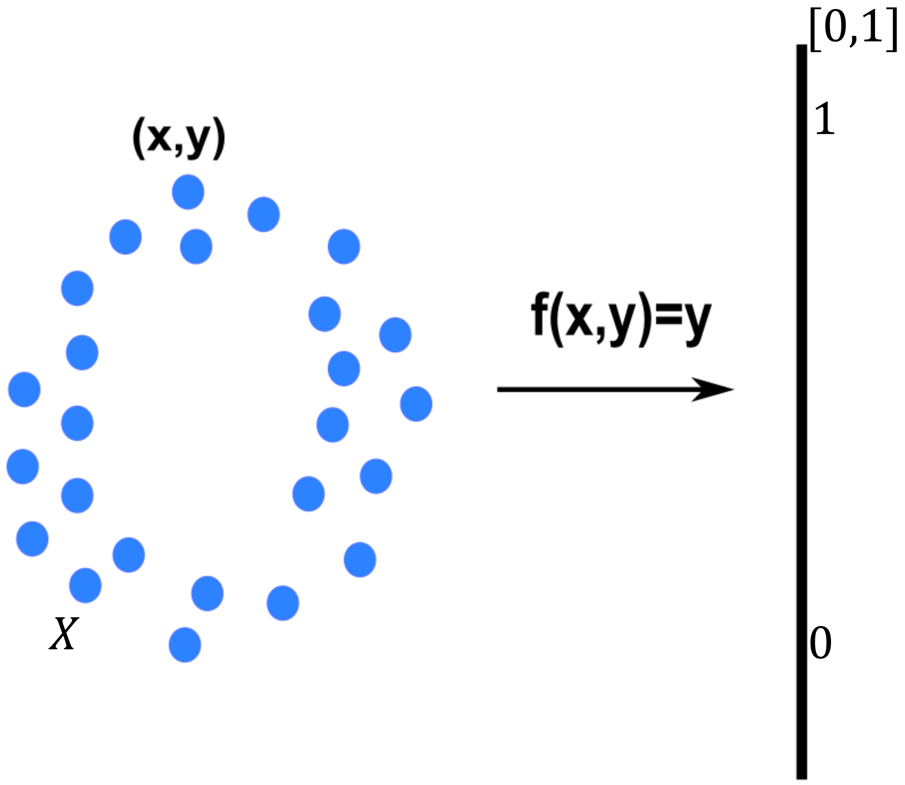


Mapper Example

Suppose that we are given a data set X and a scalar function $f: X \rightarrow [a, b]$ defined on every point in X.



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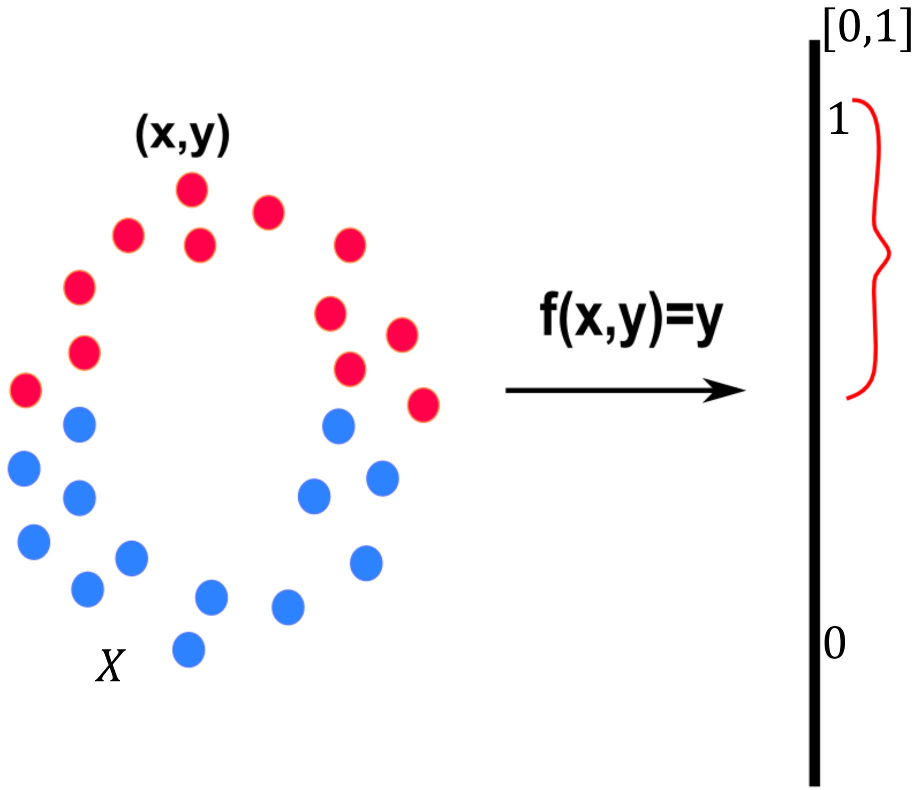

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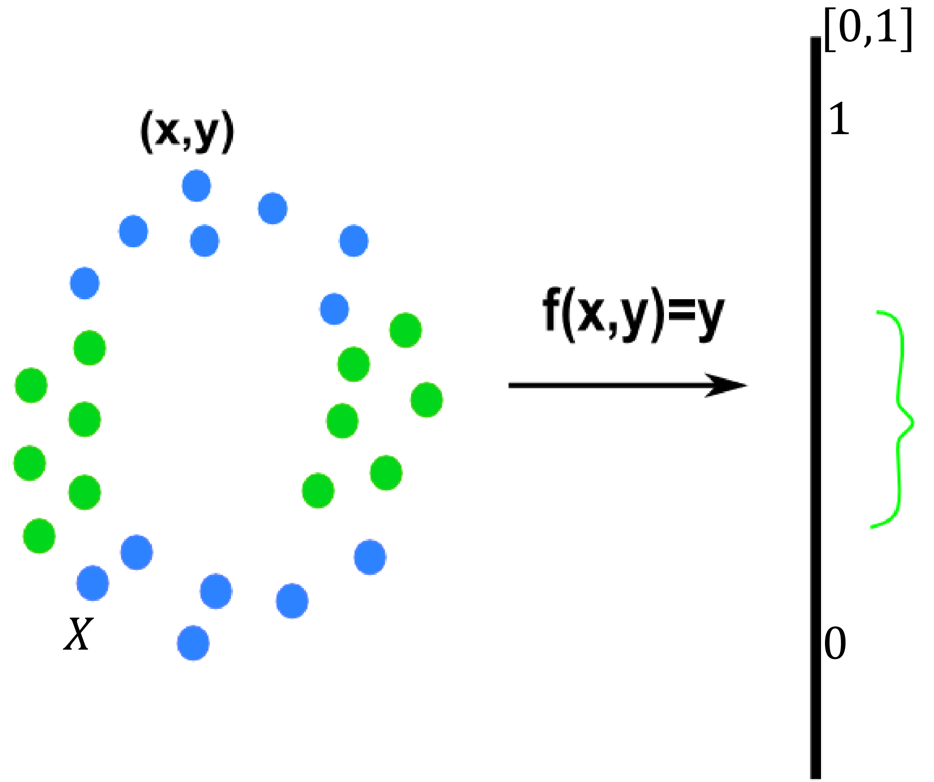

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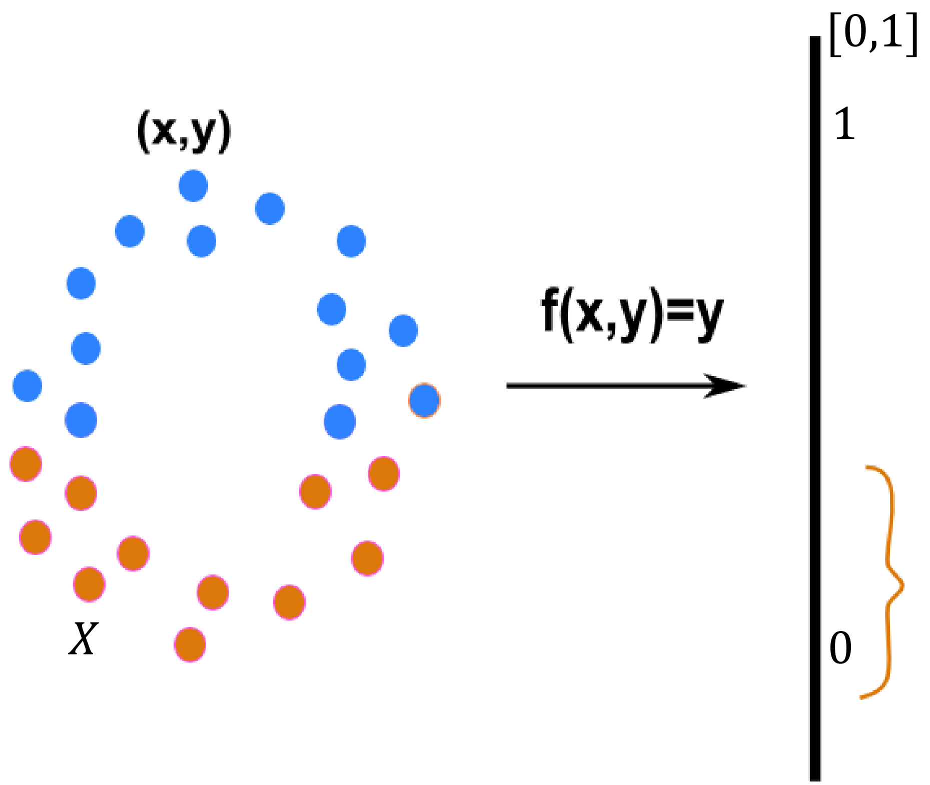

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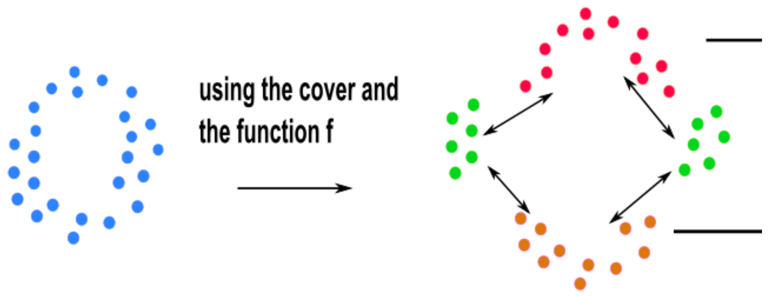

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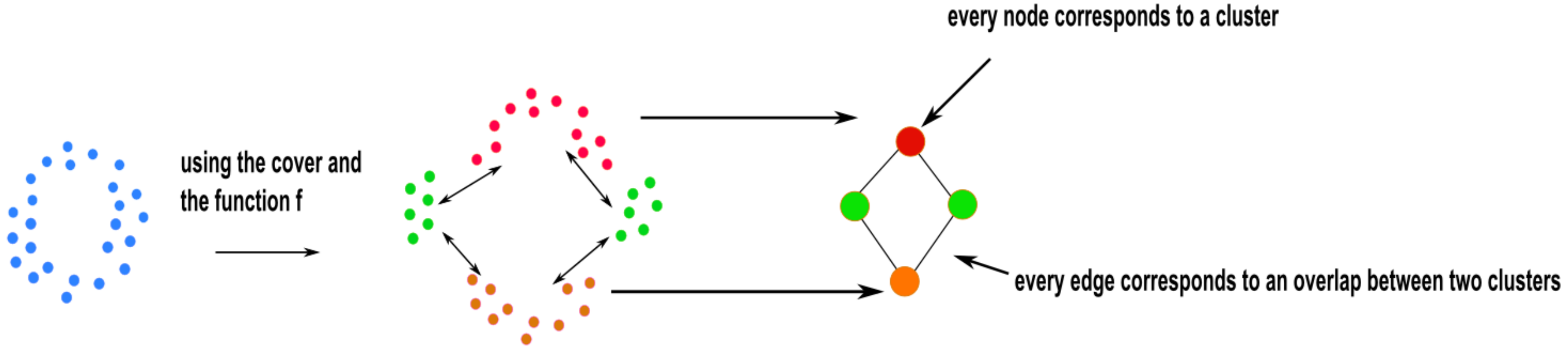
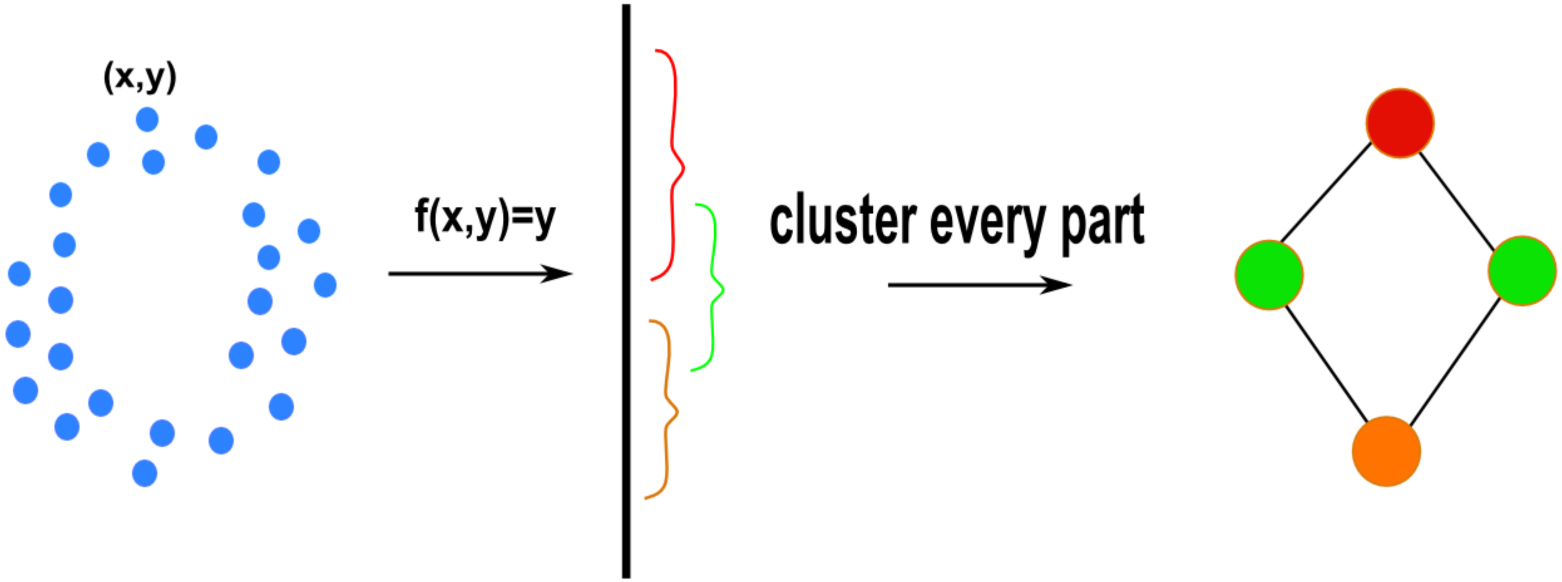
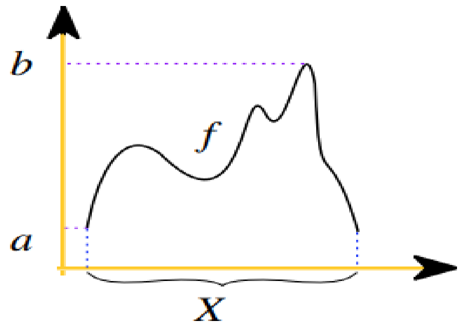


Mapper Example



The construction of Mapper on a 1d function



(a)

(a) A scalar function f : X −→ [a,b].

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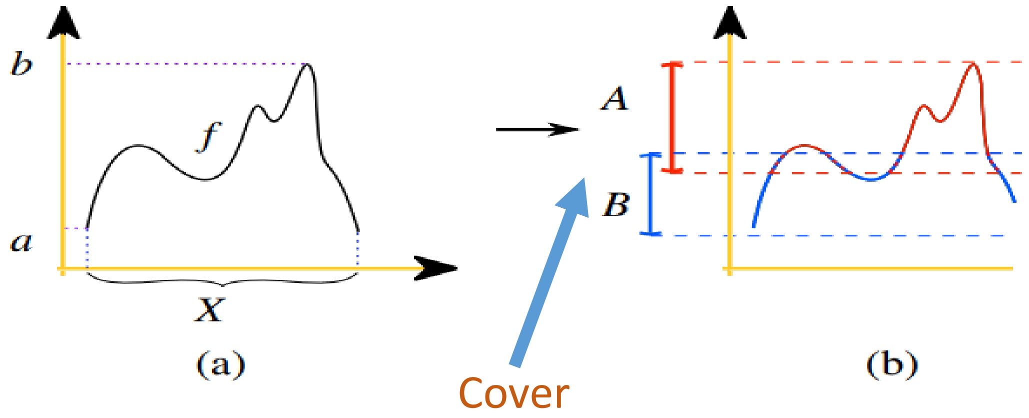

(a) A scalar function f : X –→ [a,b].

(b) The range [a,b] is covered by the two intervals A,B.

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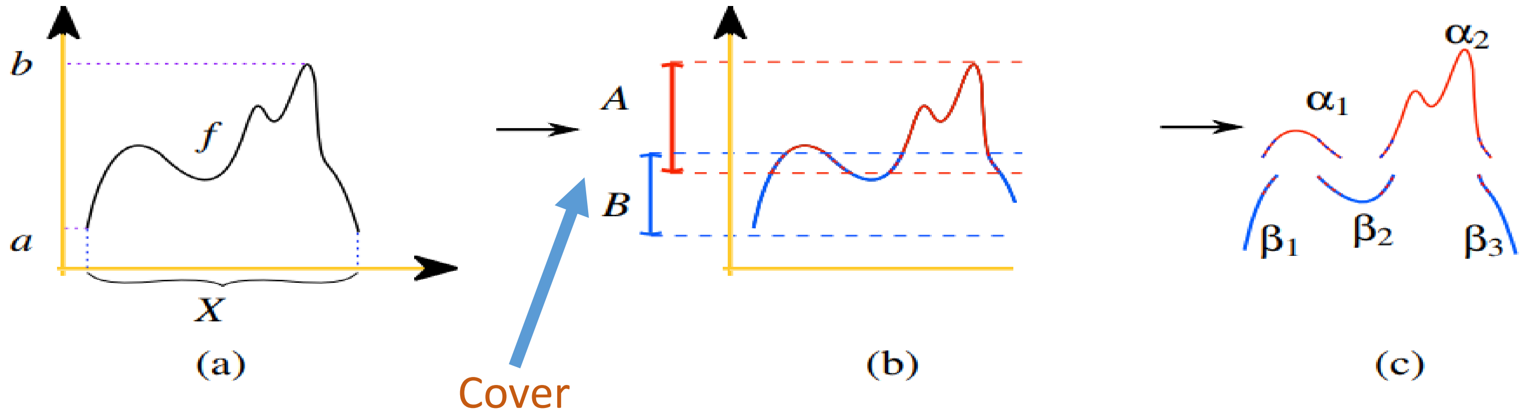

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(c) This gives a decomposition of the domain the domain X. The inverse image of A consists of two connected components α1 and α2, and the inverse image of B consists of three connected components β1, β3 and β3.

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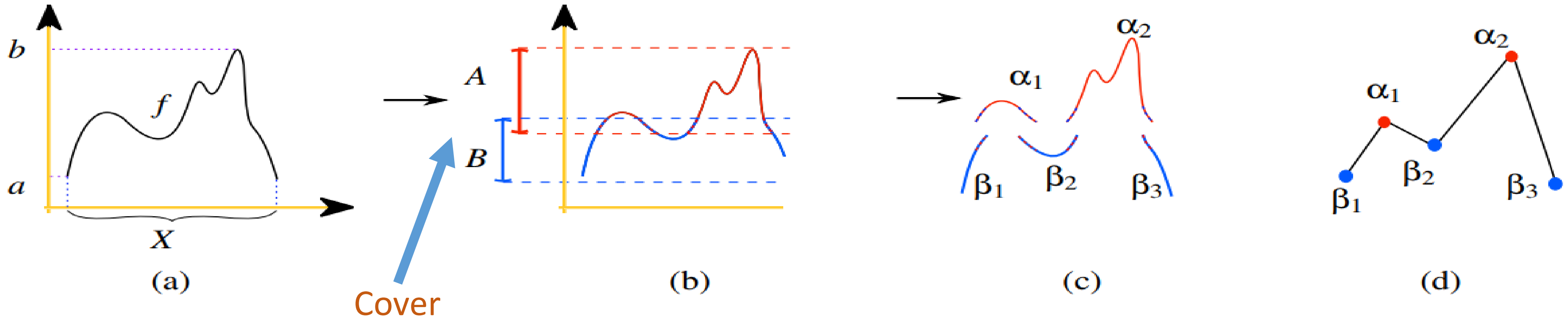

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(d) The connected components are represented by the nodes in the Mapper construction.

an edge is inserted whenever two connected components overlap.

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Suppose that we are given a data set X and a scalar function $f: X \rightarrow [a, b]$ defined on every point in X.

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