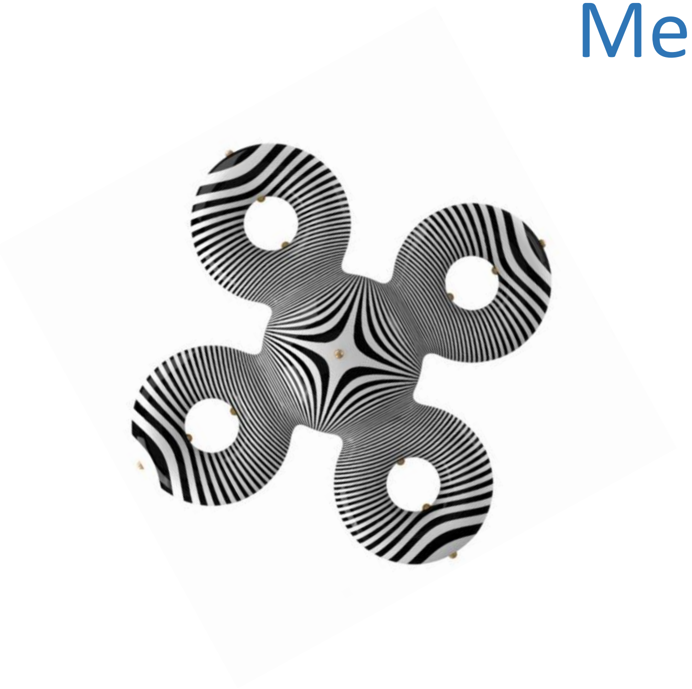
Harmonic Functions on Triangulated Meshes



Mustafa Hajij

Intuition

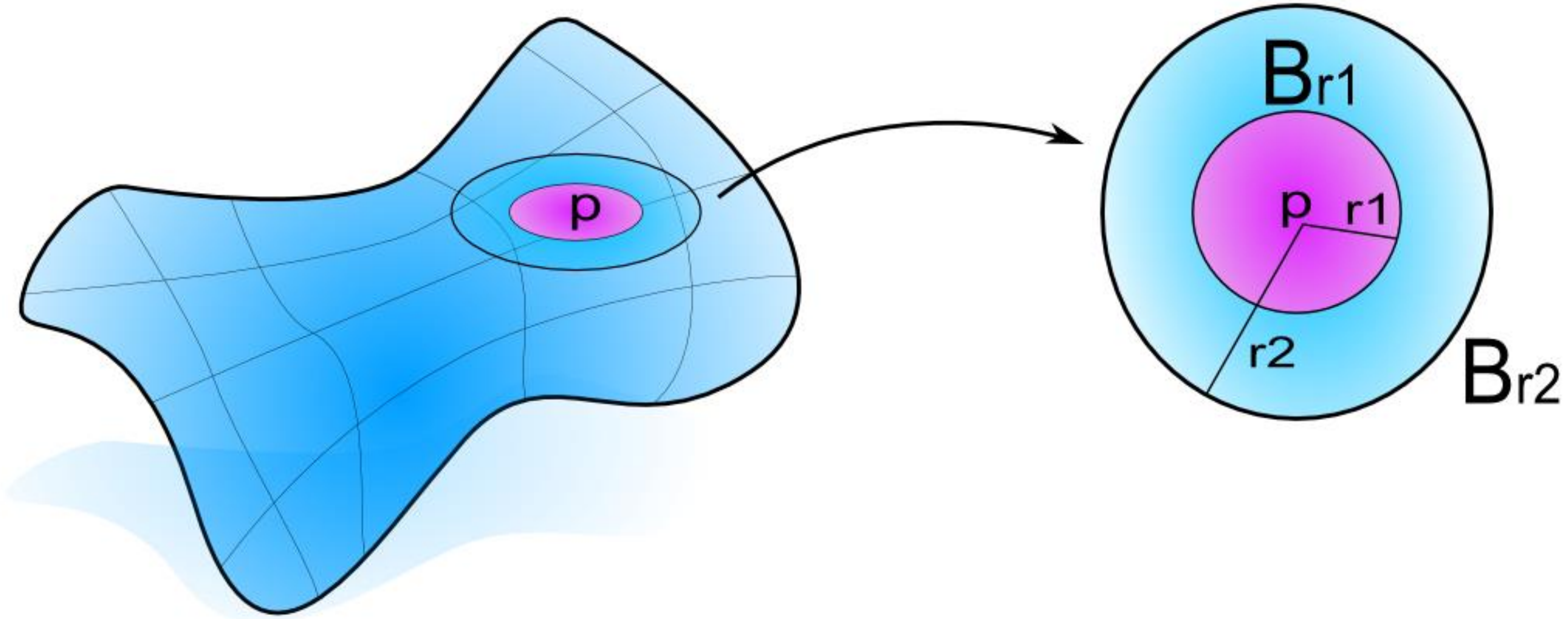
- Suppose that we have a surface made of some sort of metal.
- Suppose that we have two heat sources one of them is very hot and one of them is very cold.
- Place the hot source on one point of the metal surface and place the cold surface on another point.
- After waiting for some time the heat of the points of the surface are going to change gradually. So the points near the hot source will be also hot and the effect of this source will be less the further we move away from the source. The same applies for the cold source.
- Wait until the heat of the surface stabilizes and points on the surface are not changing its heat anymore.
- We obtain a “distribution of the heat” on the metal surface.
- The equilibrium state is very special and it has a name. We call such a “distribution” a *harmonic function* on the surface.

The Laplacian on a surface

Suppose that f is a scalar smooth function on the surface M. In other words

$$f : M \to \mathbb{R}$$

The Laplacian $\Delta f(p)$ of a function f at a point p is the rate at which the average value of f over disks centered at p, deviates from $f(p)$ as the radius of the disks grow



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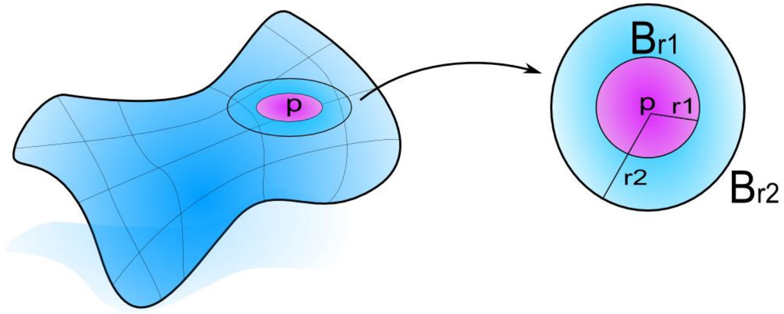
Let p be a point on the surface M. and let $B(p, r_1)$ and $B(p, r_2)$ be two disks with centered around p such that $r_1 < r_2$.



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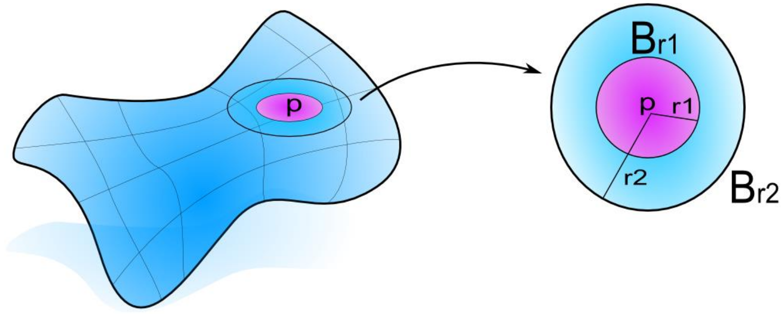
$$\int_{\partial B(p,r_1)} f\,ds$$



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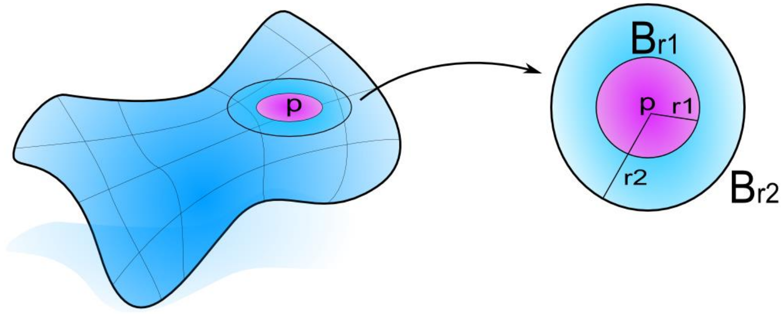

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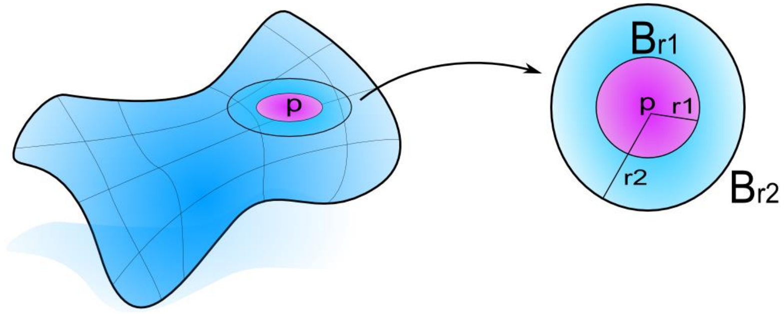

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By our earlier intuition about the Laplacian $\Delta f(p)$, this quantity is equal to the rate at which the average value of f over disks centered at $f(p)$, deviates from p as the radius of the disks grows, so this quantity approximately by

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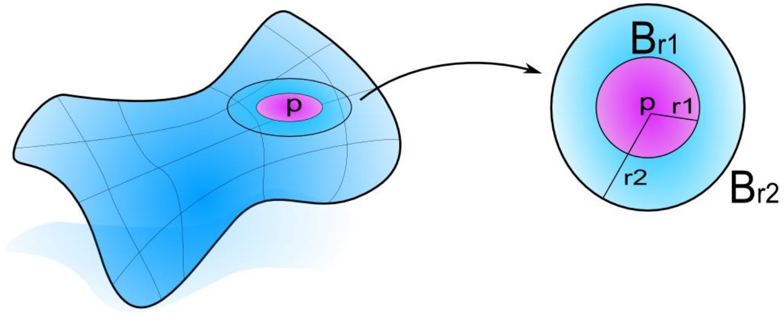

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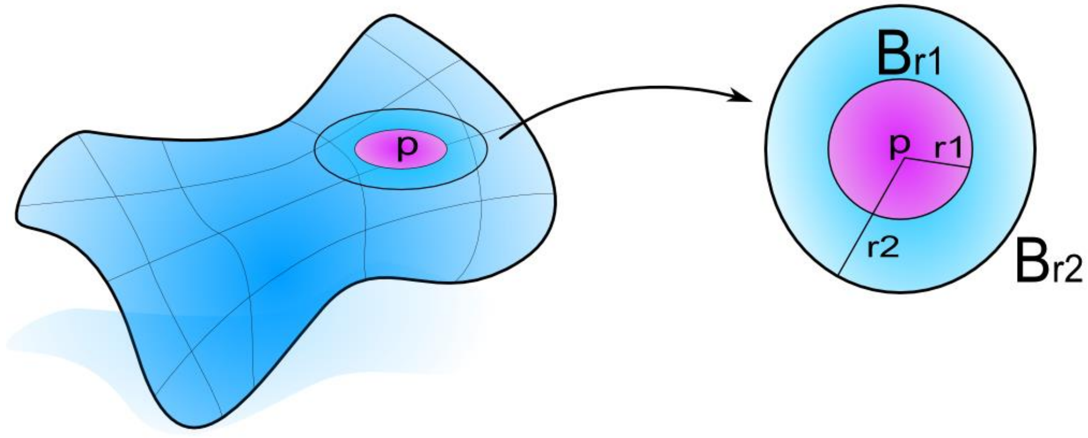
$$\Delta f(p) \simeq \int_{\partial B(p,r_2)} f ds - \int_{\partial B(p,r_1)} f ds$$

Harmonic Functions

A function $f : M \to \mathbb{R}$ is harmonic if the rate of change $\Delta f(p)$ at every point p does not change. In other words a function f is harmonic if $\Delta f(p) = 0$ or



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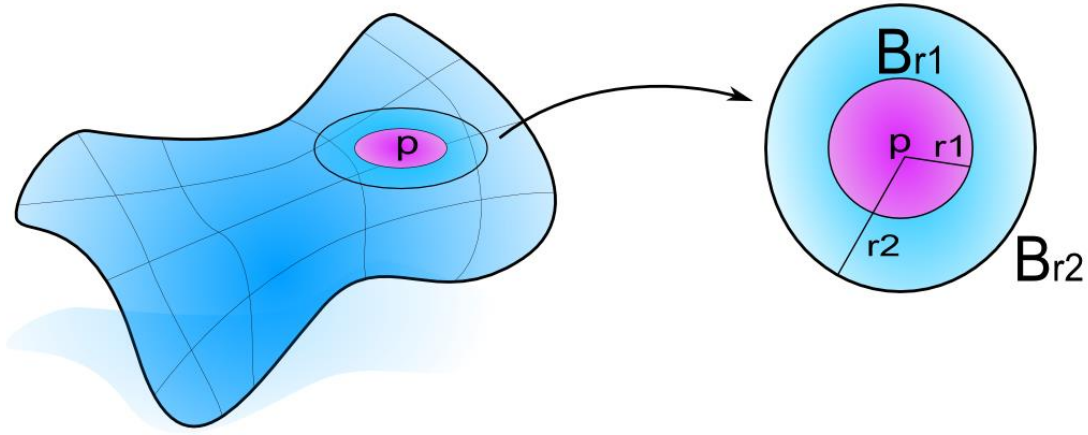

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hence a function f is harmonic if

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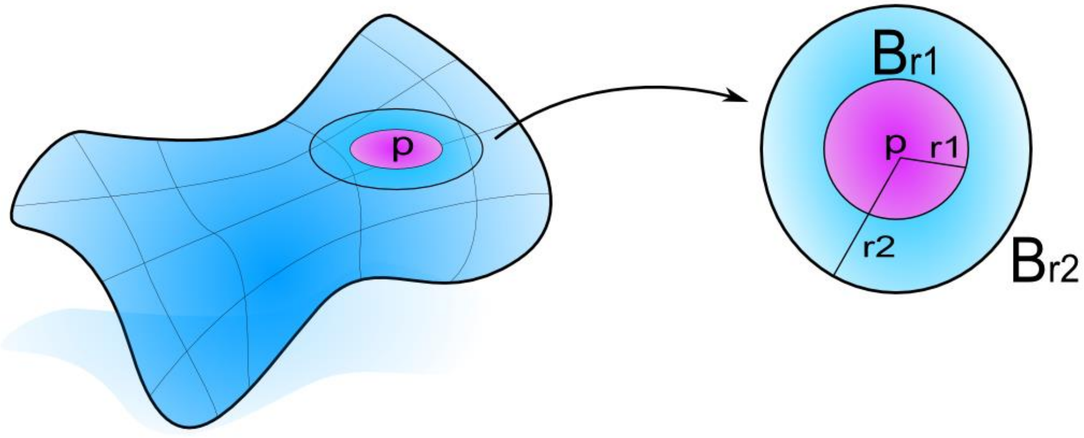

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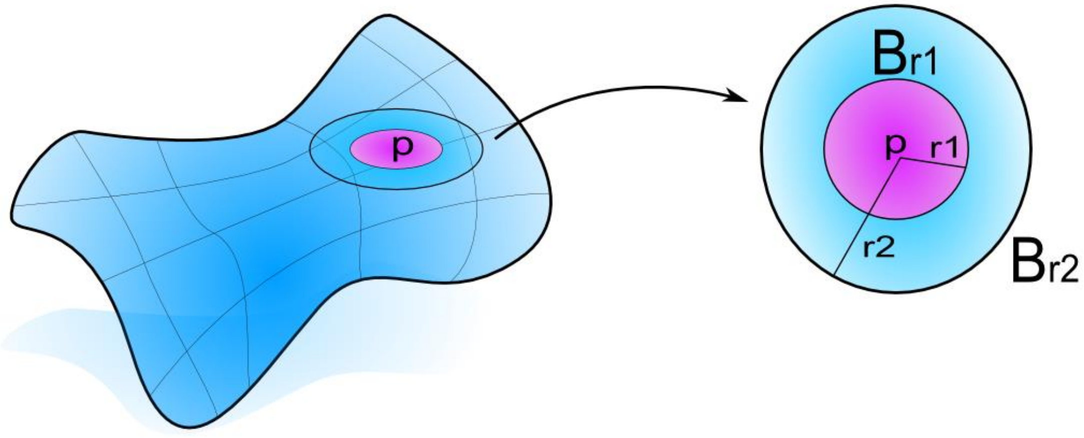

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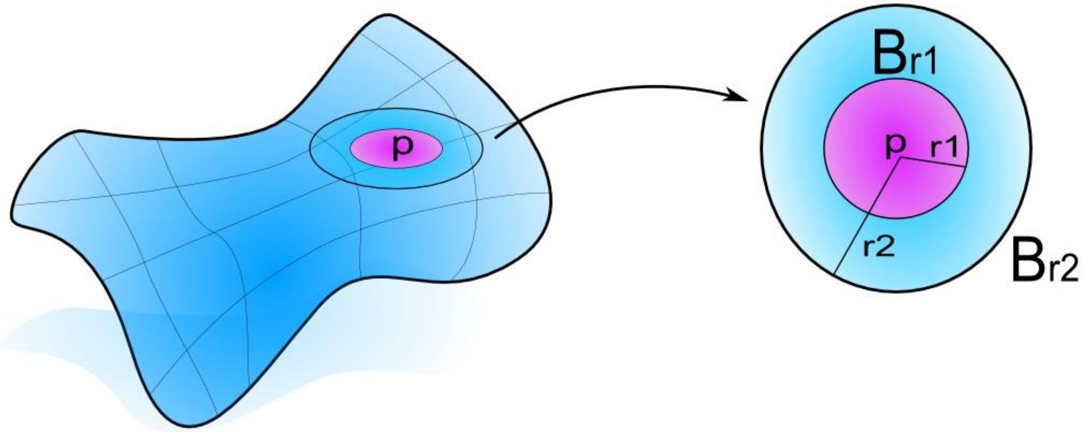

Now if we let $r_1 \to 0$ we obtain :

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so a function is harmonic if $f(p)$ is equal to the average of the summation values of the boundary of its surrounding neighborhood.

Harmonic Functions and Mean value property

This property is a very important property of harmonic functions. In fact it characterizes harmonicity:

Theorem *(Mean Value property) A function $f : M \to \mathbb{R}$ is harmonic if and only if*

$$f(p) = \frac{1}{2\pi r} \int_{\partial B(p,r)} f\,ds$$

for each point $p \in M$ and each $B(p,r) \subset M$.

Linking our intuition with the more precise concepts

- So how do we use our knowledge that we have now to write the problem that we considered in earlier in a mathematical precise way?
- So at the equilibrium state we have the following:
  1. The rate at which the average value of *f* over disks centered at *p*, deviates from *f(p)* as the radius of the disks grows.
  2. The function always has two fixed points : the hot source *p1* and the cold source *p2. Suppose that f(p1)=c1 and f(p2)=c2.*

- More precisely this can be written as follows:

$$\Delta f = 0 \text{ on } M \backslash \{p_1, p_2\} \text{ and } f(p_1) = c_1 \text{ and } f(p_2) = c_2$$

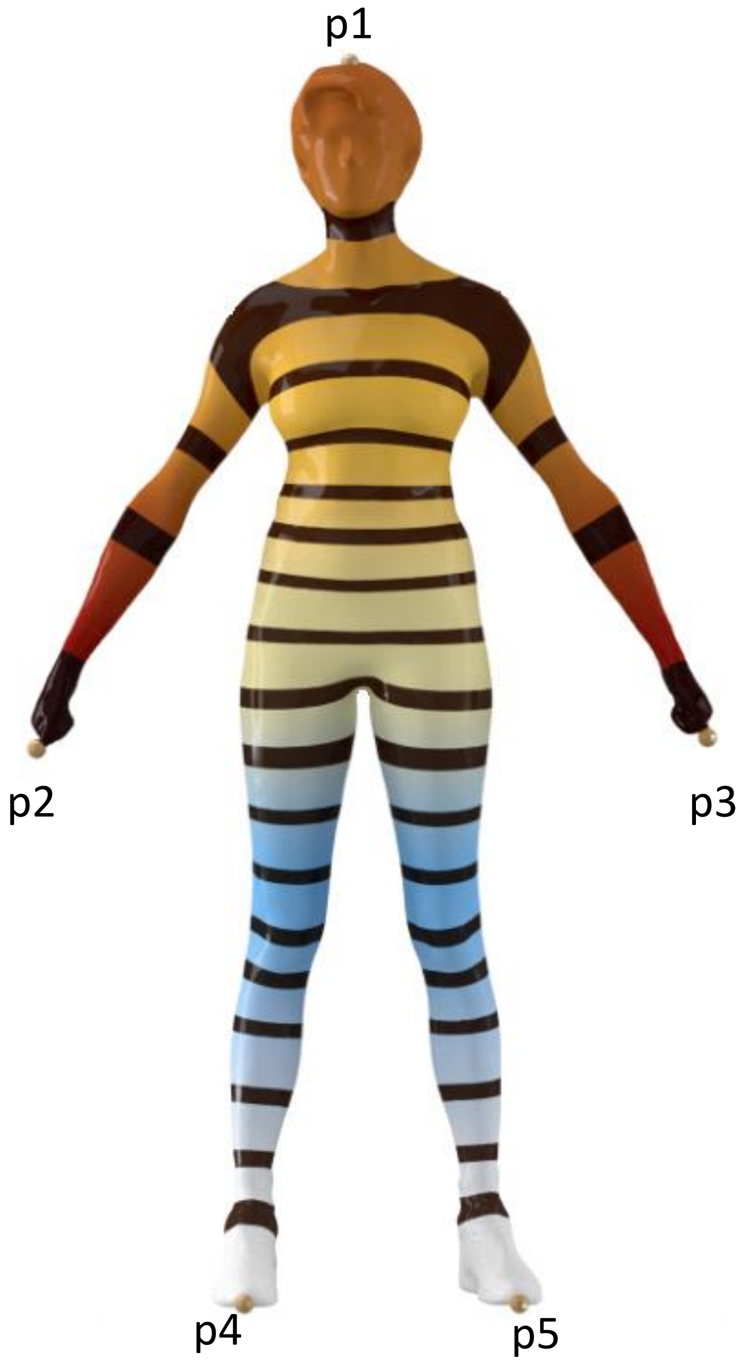
Laplace equation

Such an equation is called the Laplace equation. The equation has the following more general form :

$$\Delta f = 0 \text{ on } M \backslash D \text{ and } f(p) = g(p) \text{ for every point on } D$$

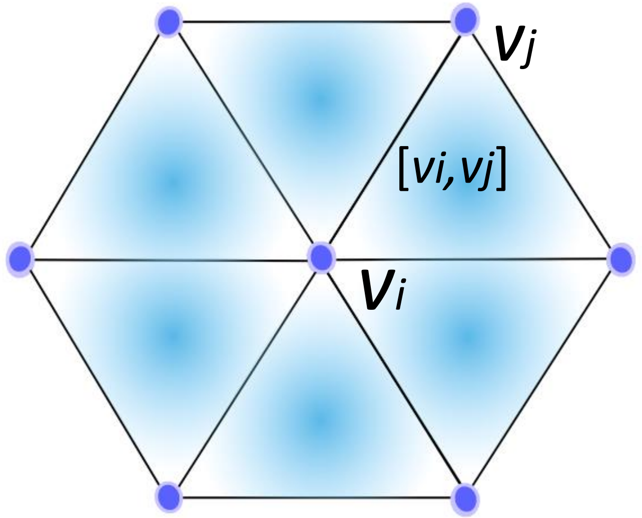
For instance, the solution in the Figure given here is obtained by solving :

$$\begin{aligned}
&\Delta f = 0 \text{ and} \\
&f(p_1) = c_1 \\
&f(p_2) = c_2 \\
&f(p_3) = c_3 \\
&f(p_4) = c_4 \\
&f(p_5) = c_5 \\
&\text{So here } D = \{p_1, p_2, p_3, p_4, p_5\}
\end{aligned}$$



Discretization of the Laplacian operator

- Let *M* be a mesh with a set of vertices *V*.
- Let $f : M \to \mathbb{R}$ be a scalar function on the mesh M defined on the vertices of M.
- The discrete Laplace operator on a mesh *M* of *f* at a vertex *vi* is defined as follows



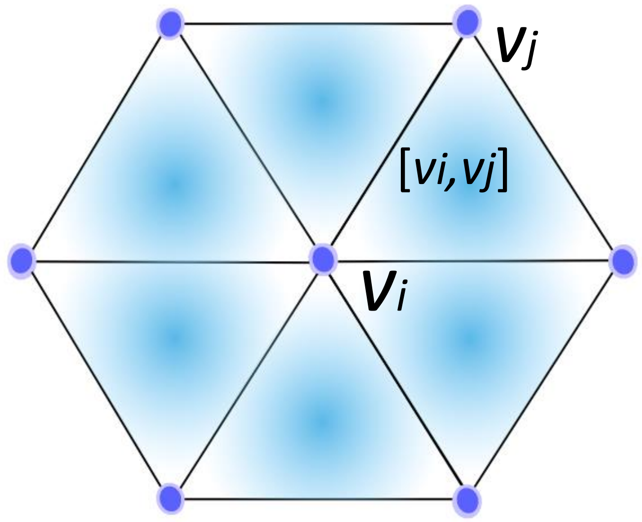
$$\Delta f(v_i) = \sum_{v_j \in lk(v_i)} w_{ij}(f(v_j) - f(v_i))$$

Where :

1. *lk(vi)* is the set of vertices adjacent to the vertex v_i
2. w_{ij} is a certain positive weight associated to the edge [*vi,vj*]. For instance w_{ij} *can be chosen to be 1 for each edge* [*vi,vj*] on the mesh *M*.

Discretization of the Laplace equation

Assume that we are provided with a set $V_C \subset V$ of $k > 1$ constrained points, where f must take on a specified values. In other words $f(v_i) = g(v_i)$ for every $v_i \in V_C$. To solve the Laplace equation, we solve the system of equations



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\begin{aligned}
\Delta f(v_i) &= \sum_{v_j \in lk(v_i)} w_{ij}(f(v_j) - f(v_i)) = 0 \text{ for every } \mathrm{v}_i \in V \backslash V_C \\
f(v_i) &= g(v_i) \text{ for every } v_i \in V_C
\end{aligned}
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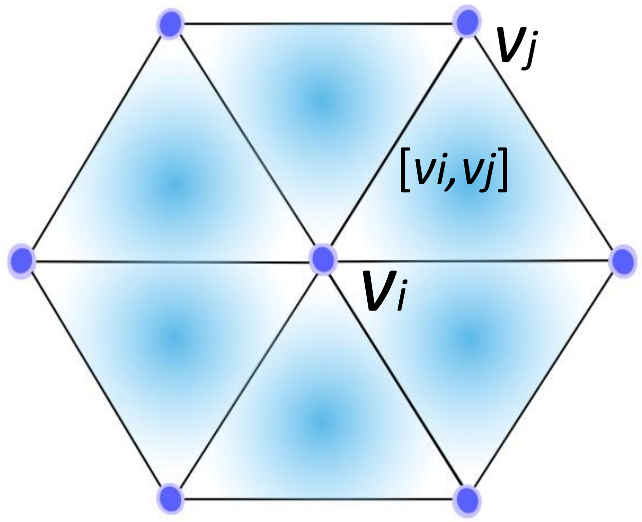

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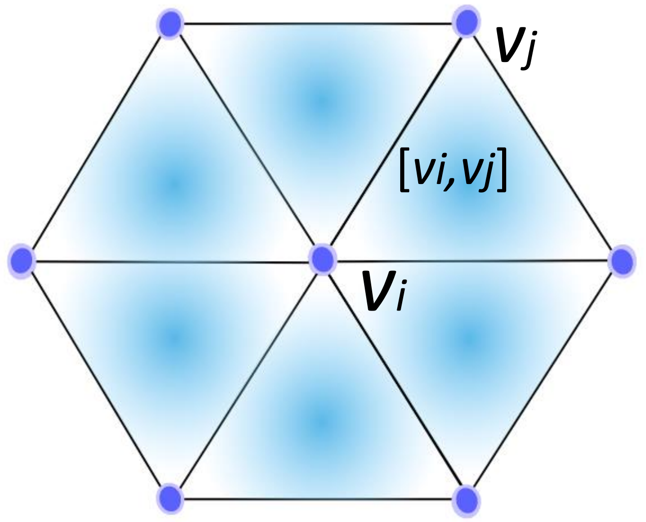

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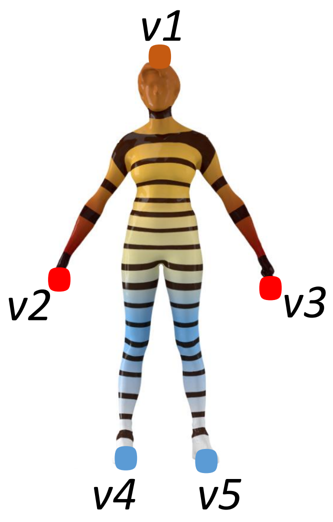
Example

Assume that $V_C = \{v_1, v_2, v_3, v_4, v_5\}$ is the set given in the Figure :



Then the harmonic function illustrated by the colors on the Figure can be obtained by solving :

$\Delta f(v_i) = 0$ for every v_i in $V \backslash V_C$

$f(v_1) = c_1$

$f(v_2) = c_2$

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Solving the Laplace equation on a mesh is solving a linear system

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Can be written as a linear system. We assemble the vertices' function values $f(v_i)$ into an n-vector $\mathbf{f}$. Then we can write the system above as the linear system :

$$L\mathbf{f} = 0$$

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where the elements of the *n*×*n* matrix *L* are given by

$$L_{ij} = \begin{cases} \sum_{\langle i,k \rangle \in M} w_{ik} & \text{if } i = j, \\ -w_{ij} & \text{if edge } \langle i,j \rangle \in M, \\ 0 & \text{otherwise.} \end{cases}$$

As well as the equations coming from :

$$f(vi) = g(v_i) \text{ for every } v_i \in V_C$$

Solving the Laplace equation on a mesh is solving a linear system

We can write this differently. Write the following matrix :

$$\hat{L}_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \in V_C, \\ 0 & \text{if } v_i \in V_C, i \neq j, \\ L_{ij} & \text{otherwise.} \end{cases}$$

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Then we solve :

$$\hat{L}\mathbf{f} = \mathbf{b}$$

where

$$b_i = \begin{cases} g(v_i) & \text{if } v_i \in V_C, \\ 0 & \text{otherwise,} \end{cases}$$