Morse Theory on Triangulated Meshes

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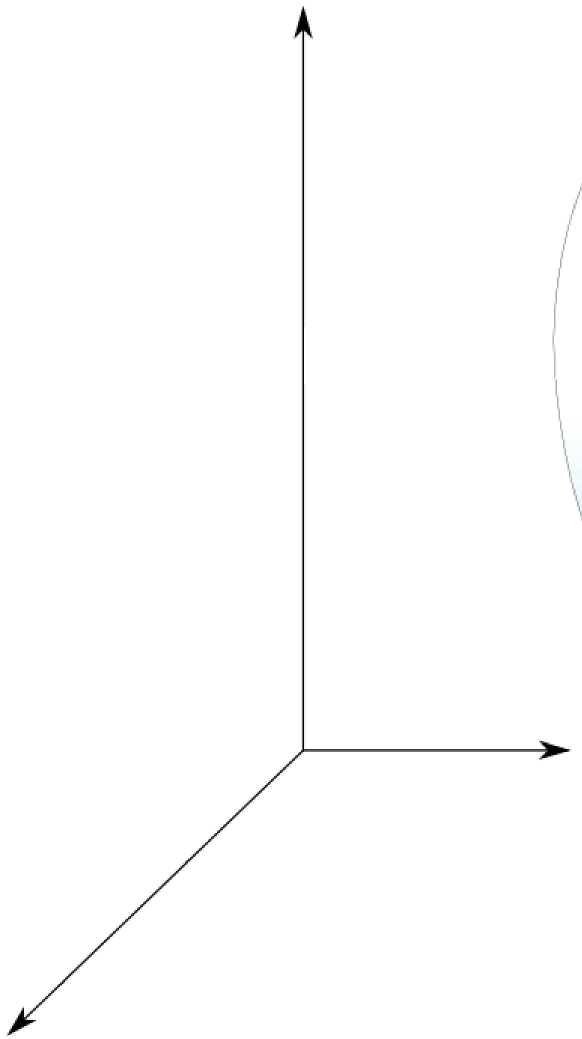
One of the primary interests in this theory is the relationship between the topology of a smooth manifold M and the critical points of a real-valued smooth function *f* defined on M.

What is Morse theory ?

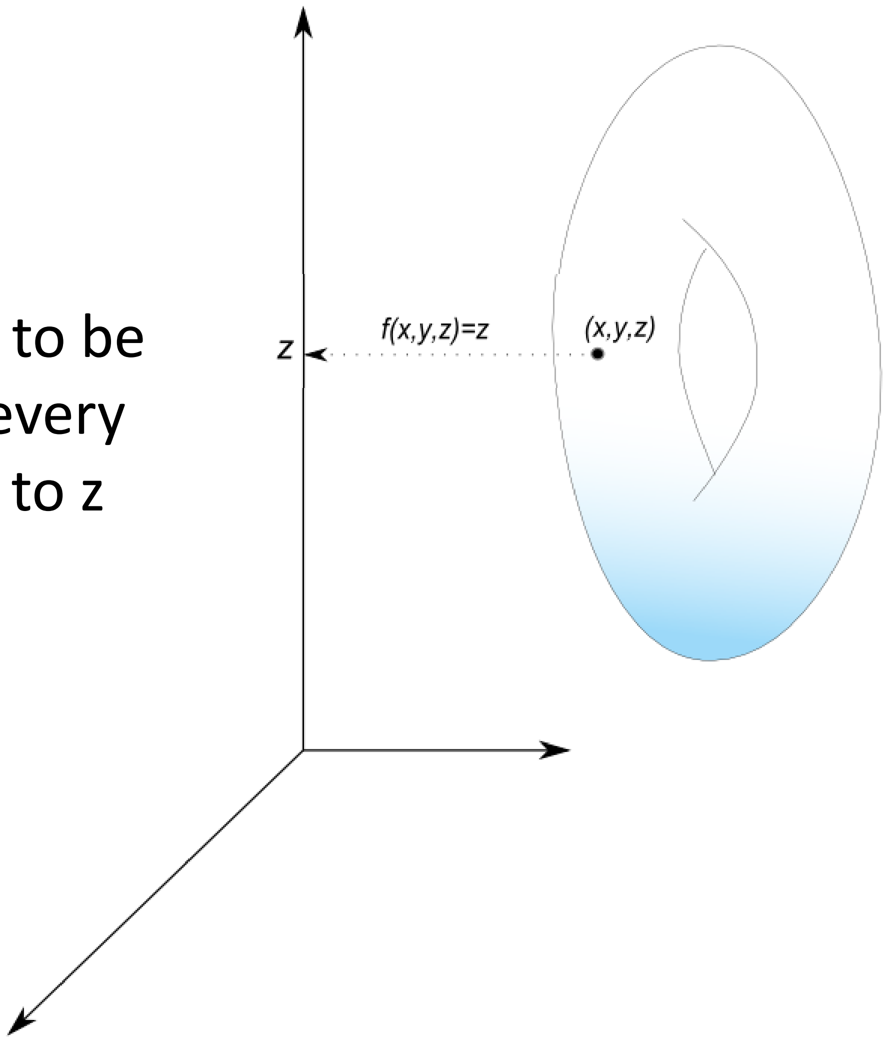
Consider the following example. Let M be a 2-dimensional torus.



What is Morse theory ?

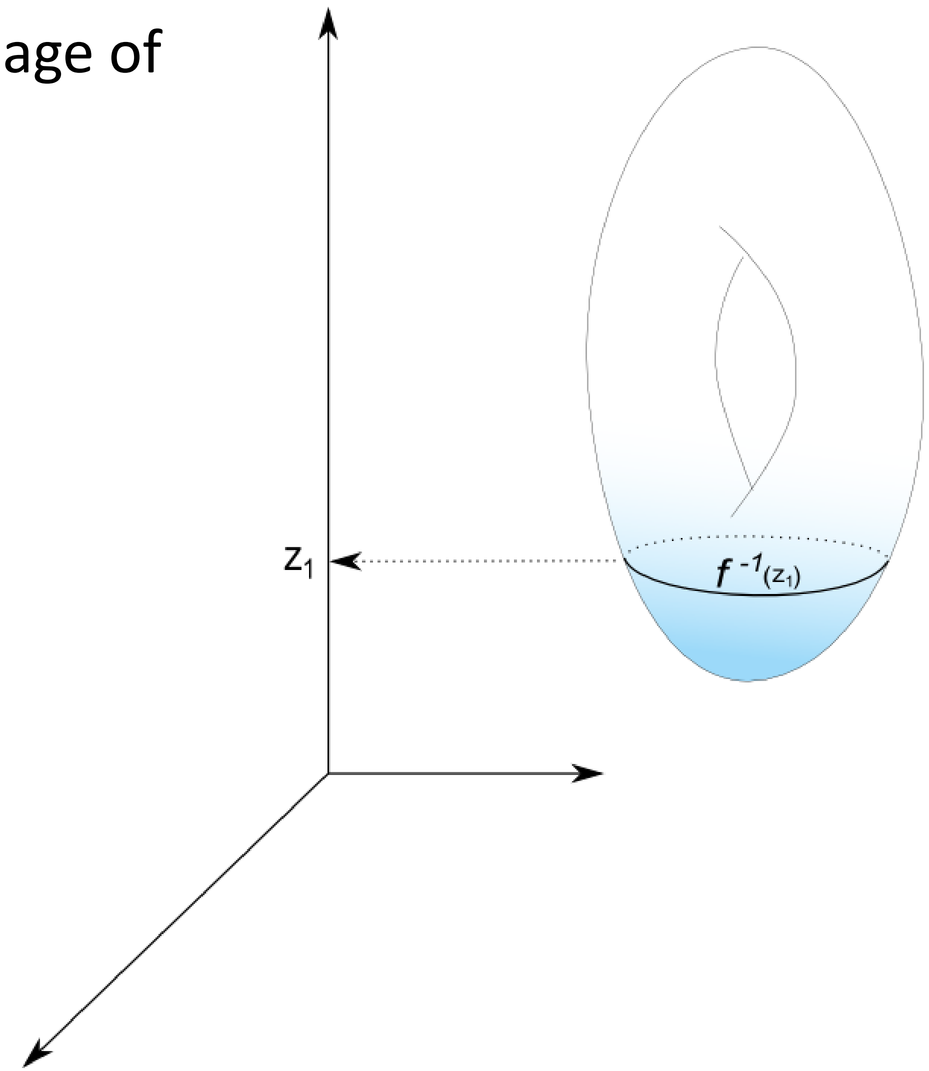
Consider the following example. Let M be a 2-dimensional torus.

Define a function f on M to be the function that sends every point (x,y,z) on the torus to z coordinate.



What is Morse theory ?

Consider the inverse image of the value z_1 under the function f.



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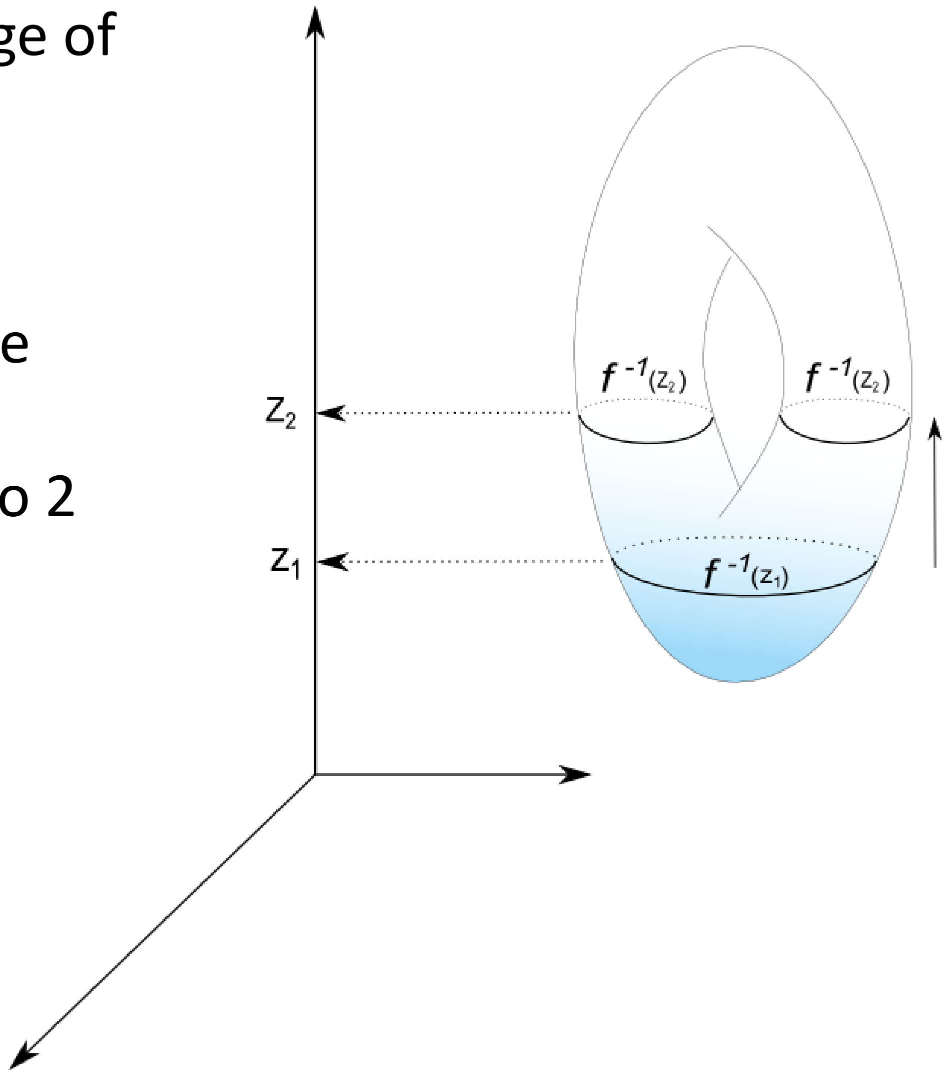
Consider the inverse image of the value z_1 under the function f.

As we increase z_1 to z_2, the topology of the level set changes from one circle to 2 circles



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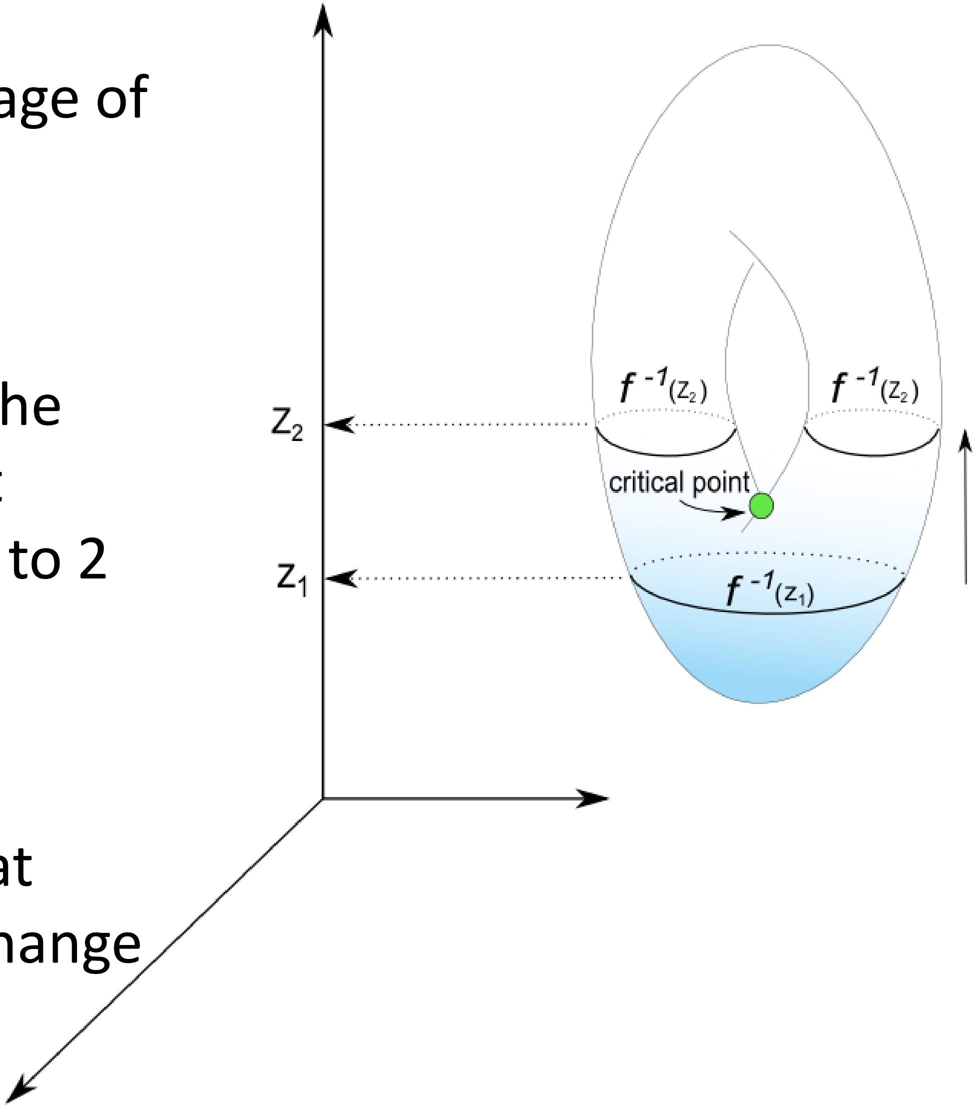
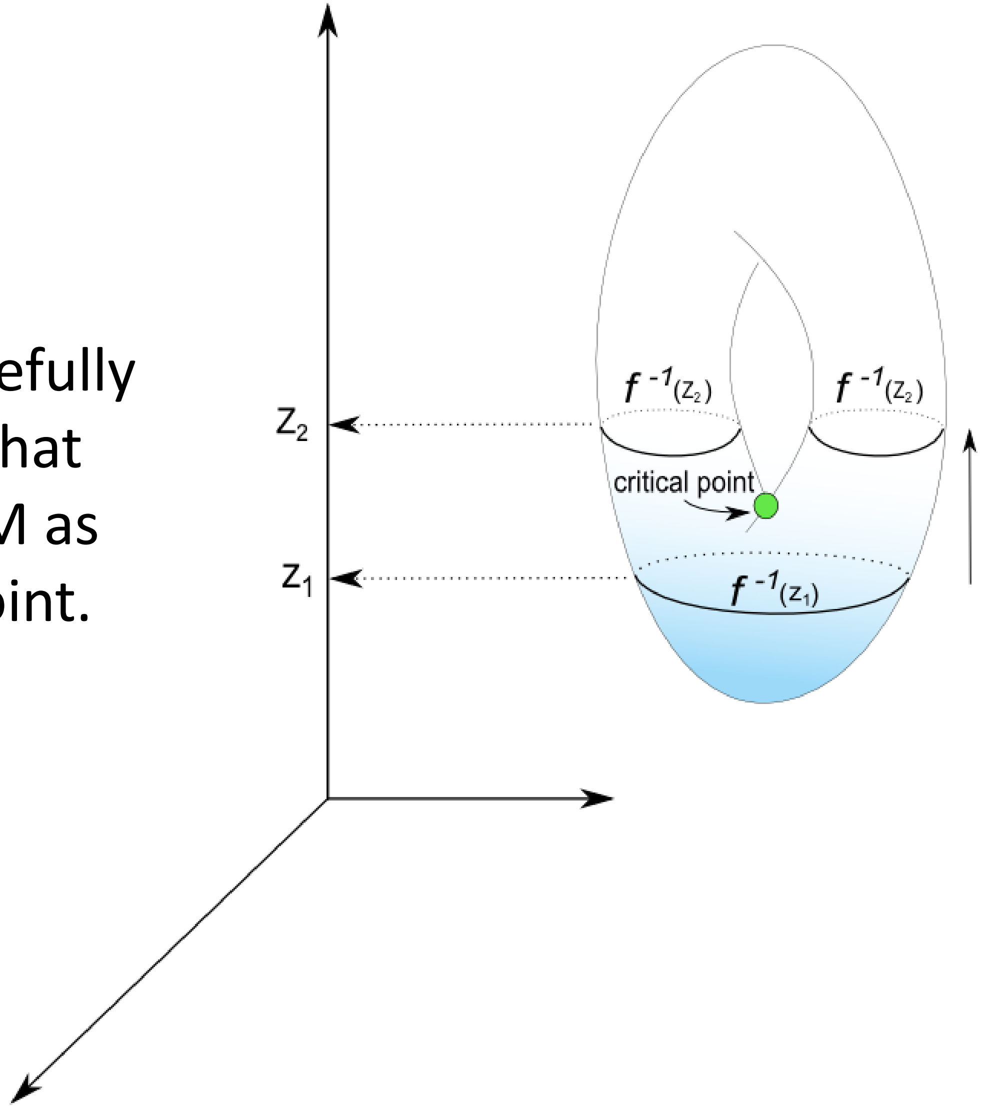
As we increase z_1 to z_2, the topology of the level set changes from one circle to 2 circles

We call the point on M at which this topological change occurs a *critical point*



What is Morse theory ?

Morse theory studies carefully the topological changes that happen to the manifold M as we pass from a critical point.



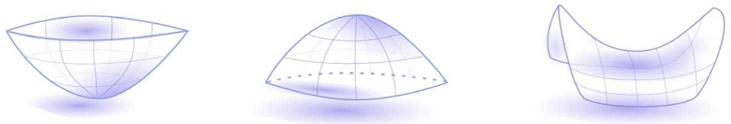
Morse theory on smooth surfaces

Let M be a compact smooth surface and let $I = [a, b] \subseteq \mathbb{R}$, where $a < b$, be a closed interval. Let $f : M \longrightarrow I$ be a smooth function defined on M. A point $x \in M$ is called a *critical point* of f if the differential df_x is zero. A value c in $\mathbb{R}$ is called a *critical value* of f is $f^{-1}(c)$ contains a critical point of f. A point in M is called a *regular point* if it is not a critical point. Similarly, if a value $c \in \mathbb{R}$ is not a critical value then we call it a regular value. The inverse function theorem implies that for every regular value c in $\mathbb{R}$ the level set $f^{-1}(c)$ is a 1-manifold, i.e., $f^{-1}(c)$ is a disjoint union of circles. A critical point is called *non-degenerate* if the matrix of the second partial derivatives of f, called the *Hessian matrix*, is non-singular. If all the critical points of f are non-degenerate and all critical points have distinct values then f is a *Morse function*.

Morse Lemma

Lemma *(Morse Lemma) Let M be a smooth surface, $f : M \longrightarrow \mathbb{R}$ be a smooth function and p be a non-degenerate critical point of f. We can choose a chart (ϕ, U) around p such that $f \circ \phi^{-1}$ takes exactly one of the following three forms:*

1. $f \circ \phi^{-1}(X, Y) = X^2 + Y^2 + c.$
2. $f \circ \phi^{-1}(X, Y) = -X^2 - Y^2 + c.$
3. $f \circ \phi^{-1}(X, Y) = X^2 - Y^2 + c.$



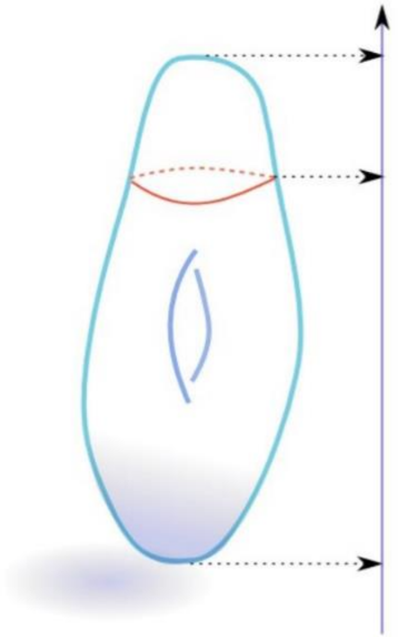
minimum saddle maximum

The form of a Morse function f around a critical point can be proven to be independent of the choice of the chart.

Handle decomposition

Let M be a smooth surface and let $f : M \longrightarrow \mathbb{R}$ be a Morse function defined on M. Define the set

$$M_{f,t} = \{x \in M : f(x) \leq t\}.$$



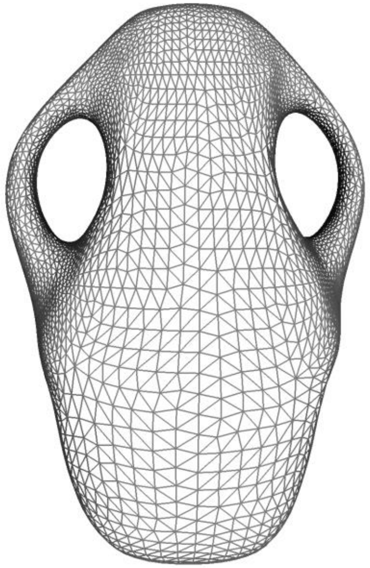
Handle decomposition

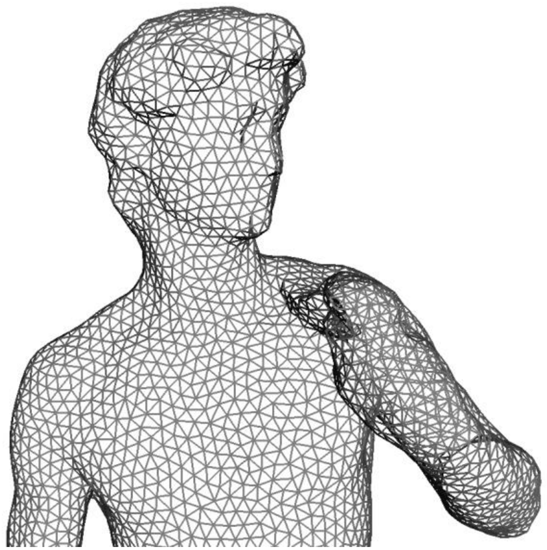
Theorem *Let $f : M \longrightarrow \mathbb{R}$ be Morse function. Let p be a critical point of index i and $f(p) = t$ be its corresponding critical value. Let ϵ be chosen small enough so that f has no critical values on the interval $[t - \epsilon, t + \epsilon]$. Then :*

1. *If $\text{index}_f(p) = 0$, then $M_{t+\epsilon}$ is diffeomorphic to the disjoint union of $M_{t-\epsilon}$ and a 2-dimensional disk D^2.*
2. *If $\text{index}_f(p) = 1$, then $M_{t+\epsilon}$ can be obtained from $M_{t-\epsilon}$ by attaching a 1-handle. That means that $M_{t+\epsilon}$ can be obtained by gluing a rectangular strip $D^1 \times D^1$ to the boundary of $M_{t-\epsilon}$ along $D^1 \times \partial D^1$.*
3. *If $\text{index}_f(p) = 2$, then $M_{t+\epsilon}$ can be obtained by capping off the surface $M_{t-\epsilon}$ with a disk D^2. That means that $M_{t+\epsilon}$ is obtained by gluing a disk D^2 along its boundary ∂D^2 to one of the boundary components of $M_{t-\epsilon}$.*

Morse Theory on Meshes

In practice we approximate surfaces by triangulated meshes.





Examples of triangulated meshes

Morse Theory on Meshes

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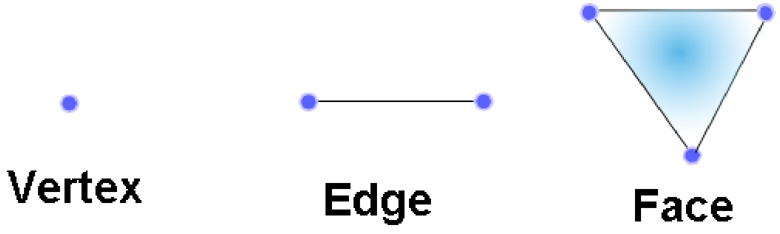
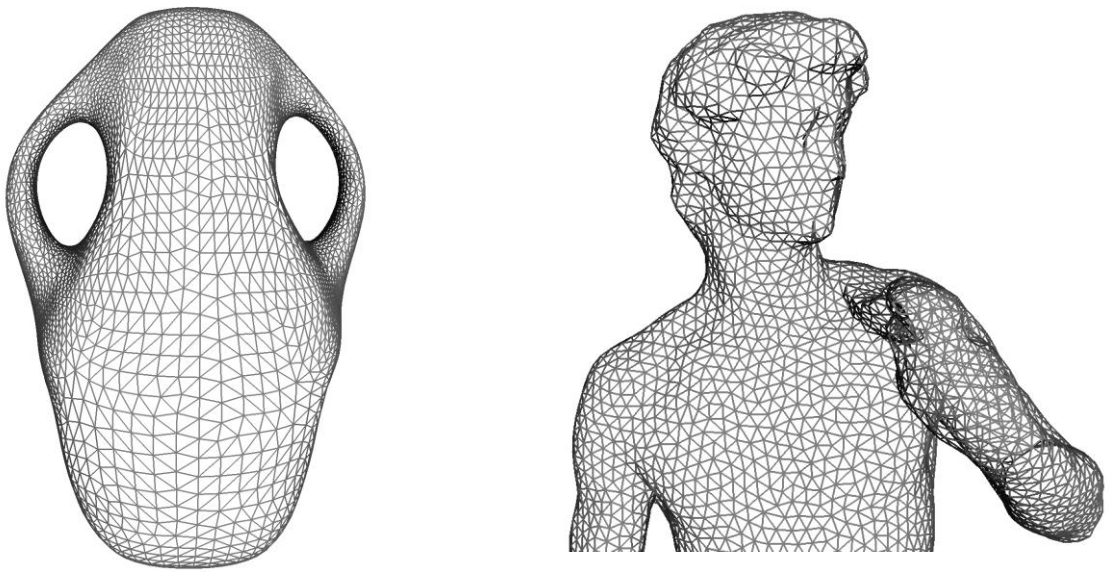


Mesh components



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Morse Theory on Meshes

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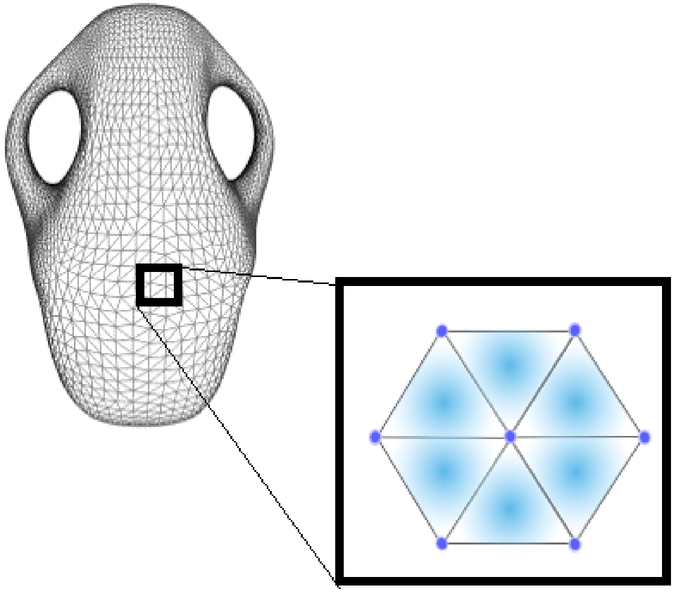

Consider a local ring neighborhood around a vertex and consider the values of v on the ring

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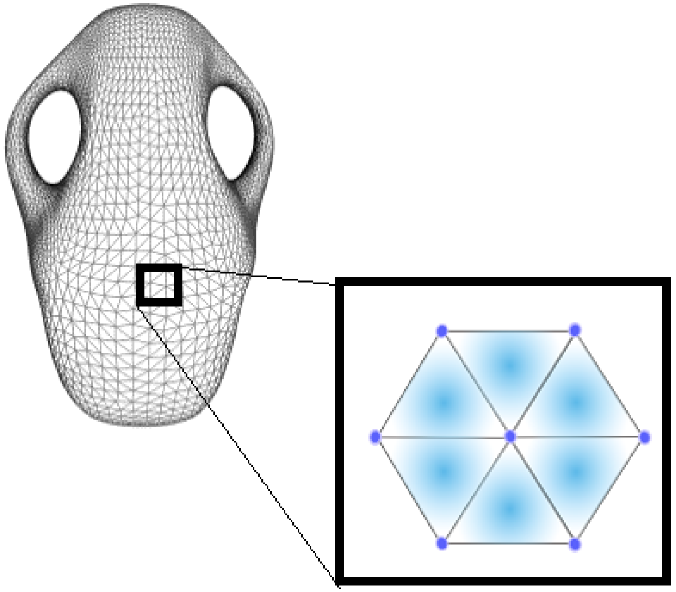

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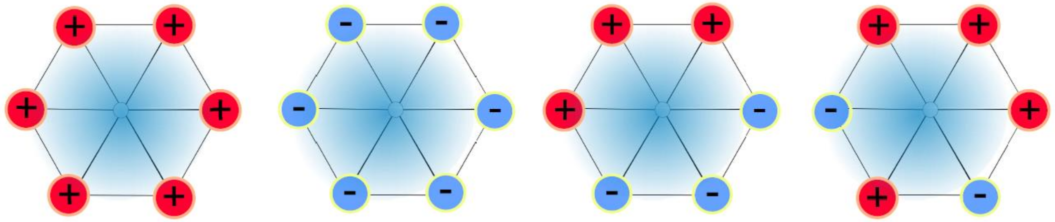
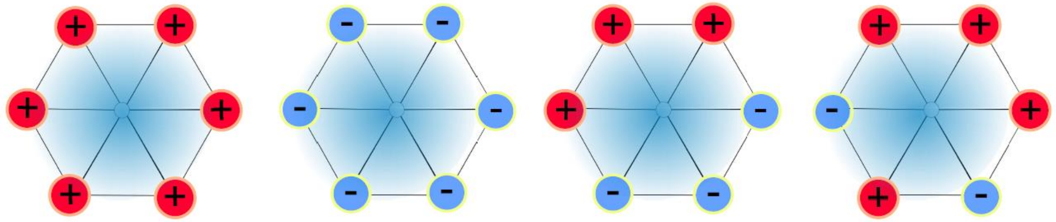


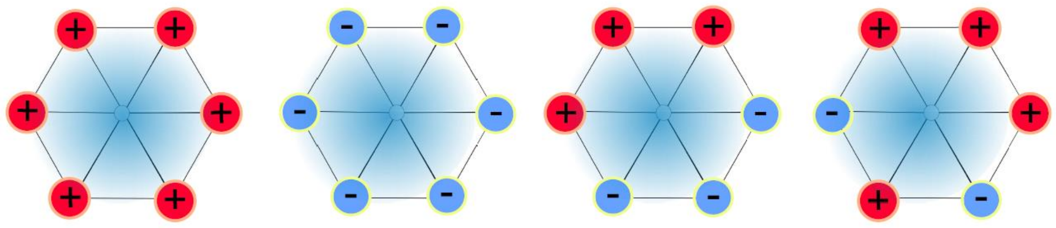
Locally, the Morse scalar function around a vertex has one of the following possibilities

Morse Scalar Functions on Meshes



minimum maximum regular saddle

Morse Scalar Functions on Meshes



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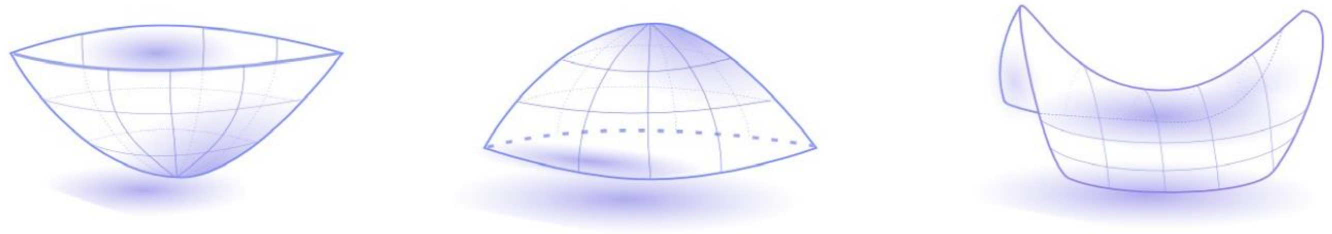


minimum maximum saddle

Types of vertices

Given a Morse function *f on a triangulated mesh M.*
Then we can classify the vertices of M as follows

The upper link of v is defined as

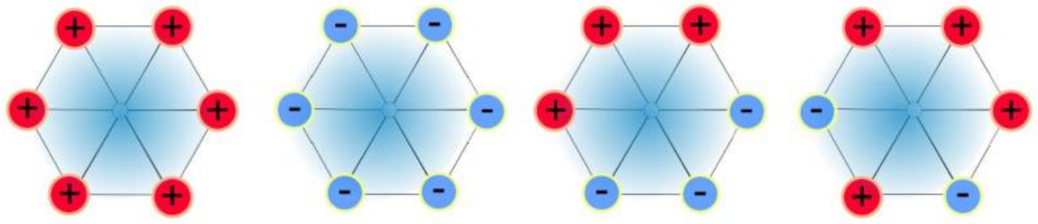
$$Lk^+(v) = \{u \in Lk(v) : f(u) > f(v)\},$$

and the lower link is defined by

$$Lk^-(v) = \{u \in Lk(v) : f(u) < f(v)\},$$

and mixed link

$$Lk^\pm(v) = \{(u_1, u_2) : f(u_1) < f(v) < f(u_2)\}.$$



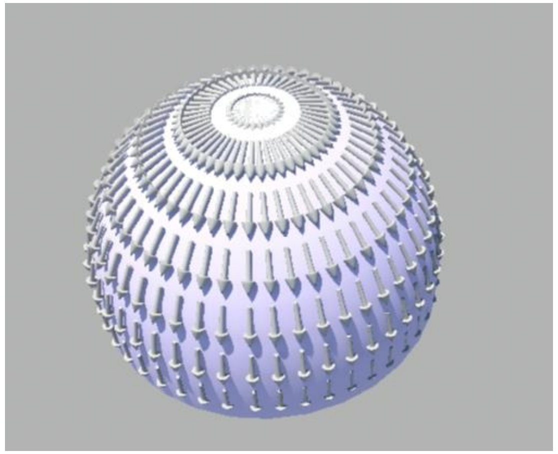
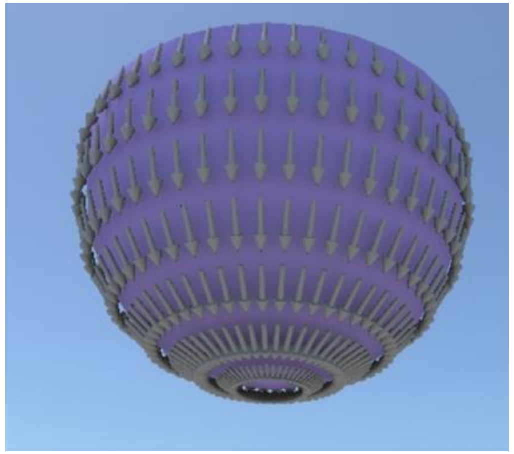
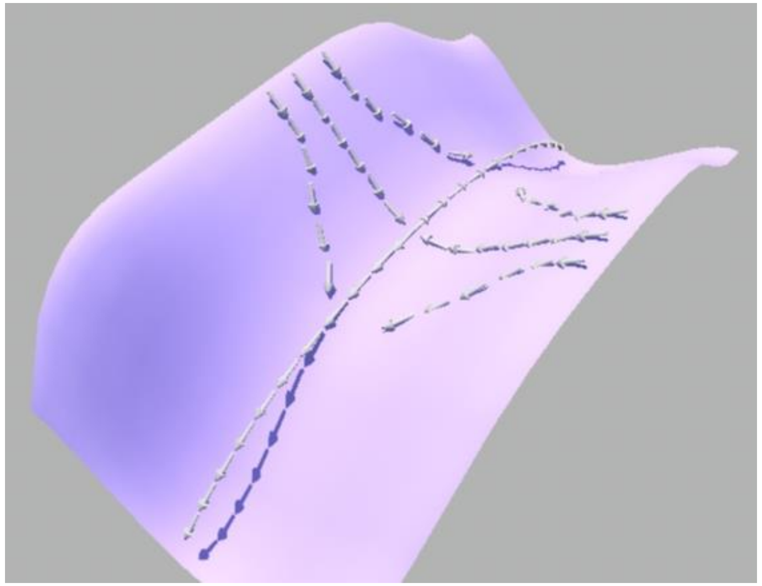
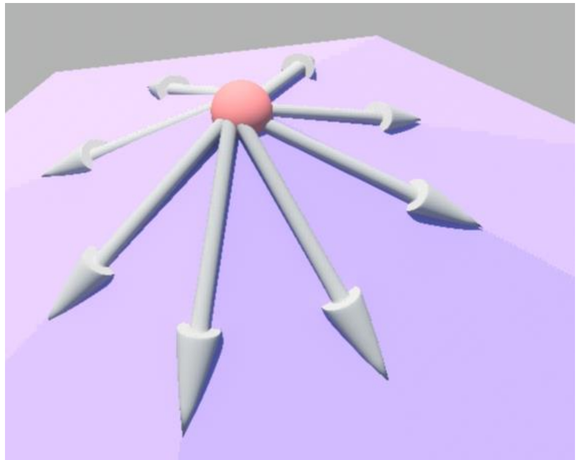
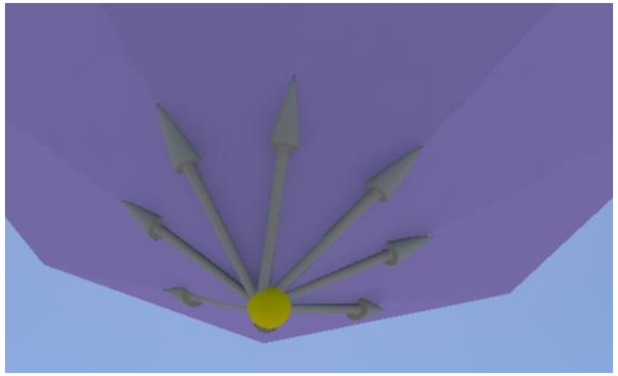
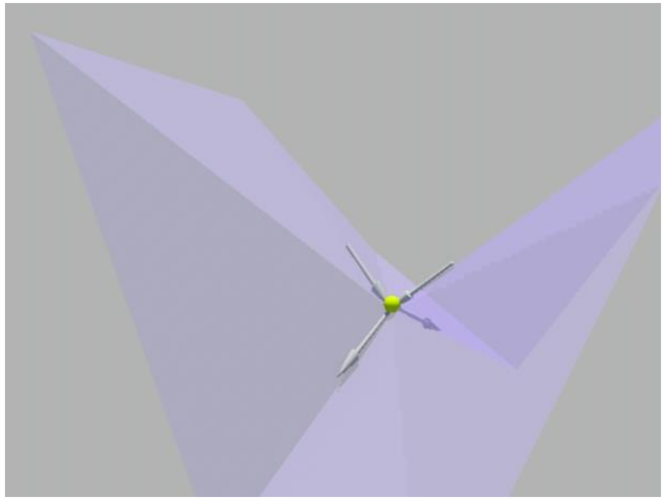
v is *regular* if $|Lk^\pm(v)| = 2$

a *maximum* with index 2 if $|Lk^+(v)| = 0.$

is a *minimum* with index 0 if $|Lk^-(v)| = 0$

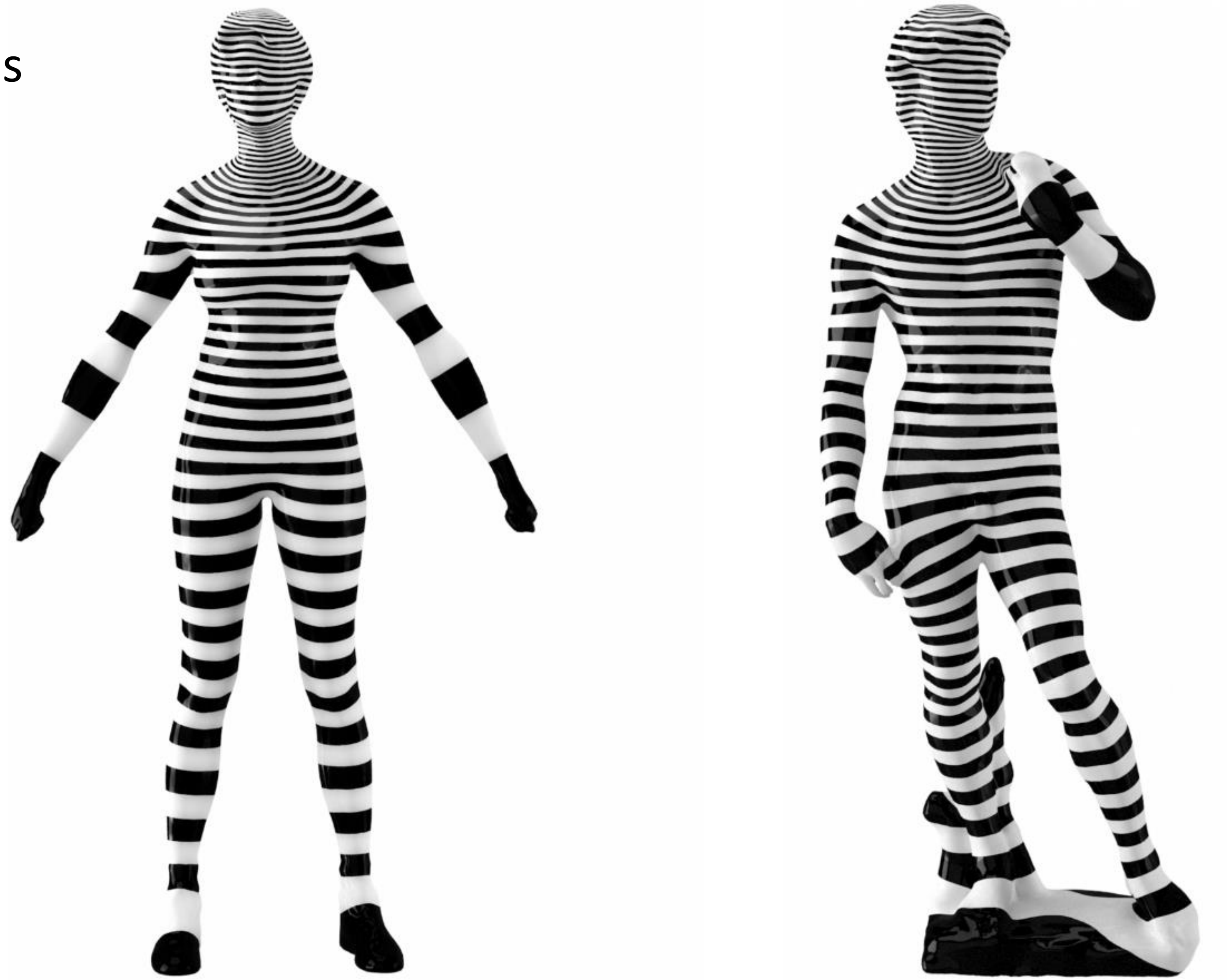
saddle with index 1 and multiplicity $m \geq 1$ if $|Lk^{\pm(v)}| = 2 + 2m$

Definition: A scalar function *f* on a triangulated mesh M is PL Morse function if each vertex is either regular or simple critical (minimum, maximum or saddle with m=1) and the function values of the vertices are distinct.

Morse Scalar Functions on Meshes

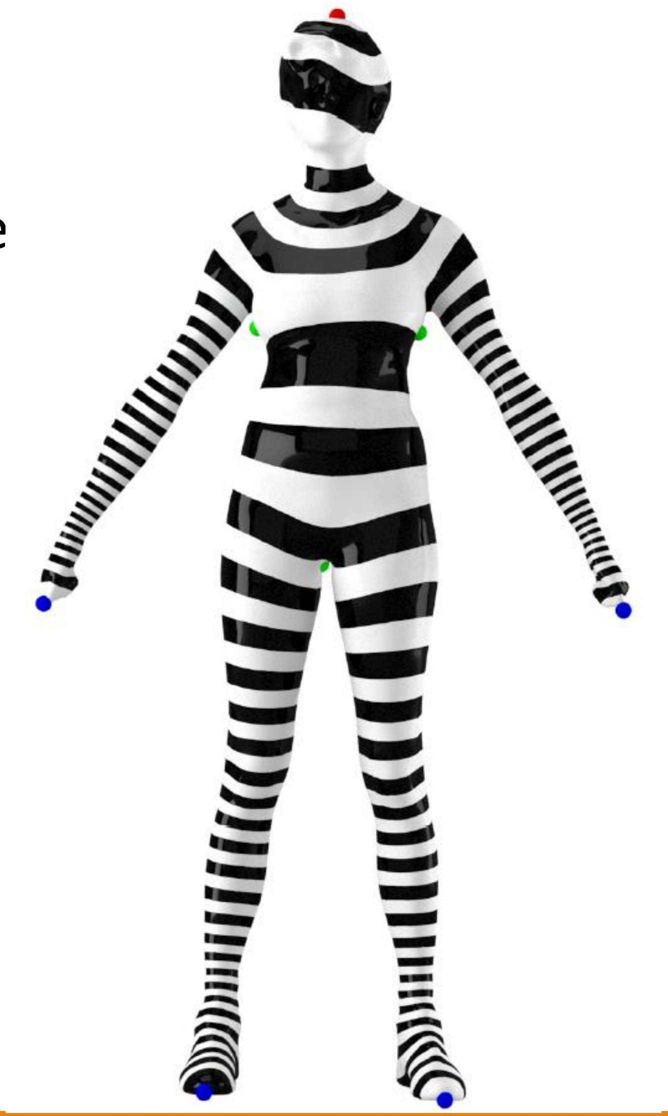
We will represent the isolines as follows:



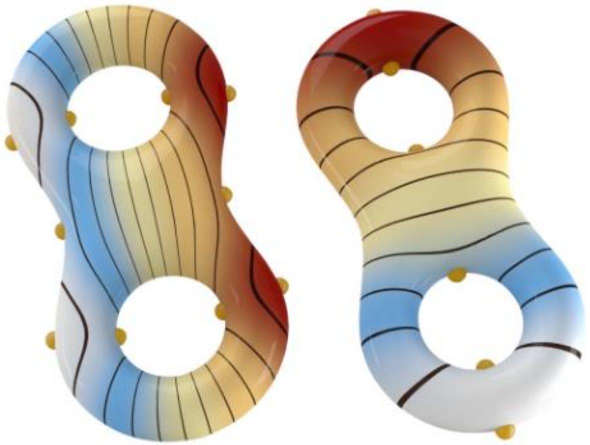
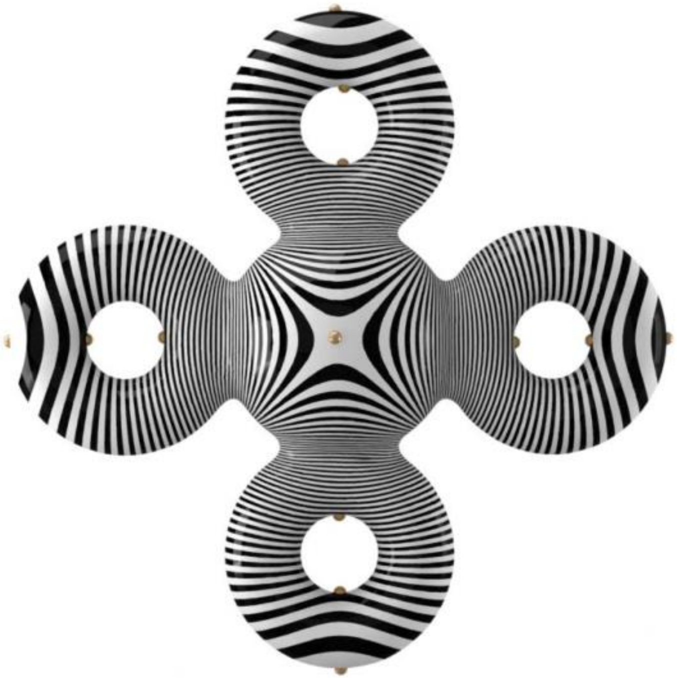
Morse Scalar Functions on Meshes

In the following example we represent :

minimum point by a blue sphere
maximum point by a red sphere
saddle point by a green sphere



Morse Scalar Functions on Meshes


Applications of Morse Theory

Morse theory has found many applications recently in geometric processing

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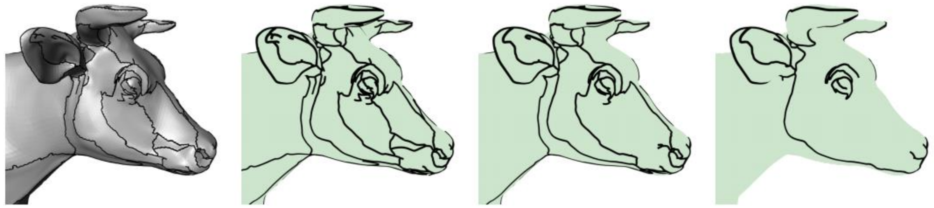
Extracting Surface feature lines (Sahner et. el.)



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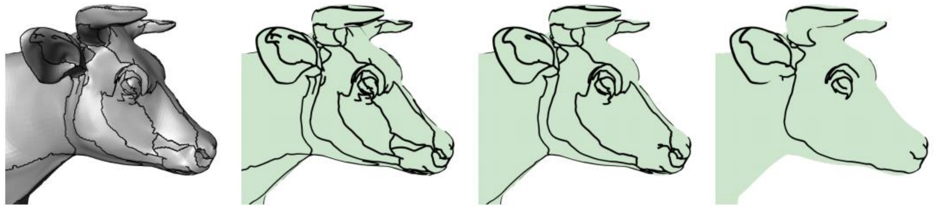

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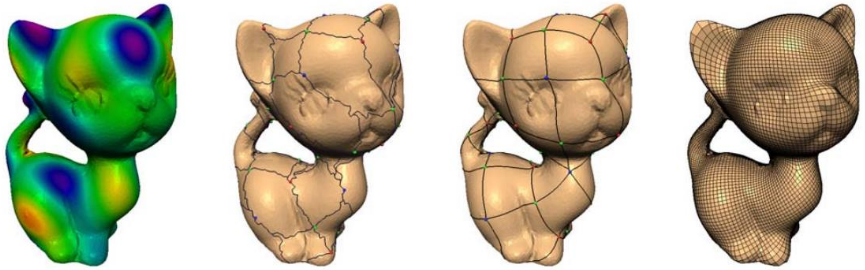

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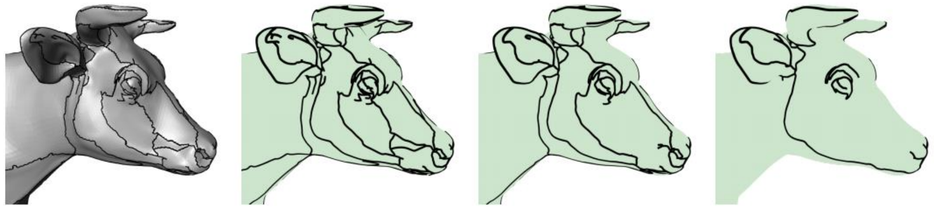

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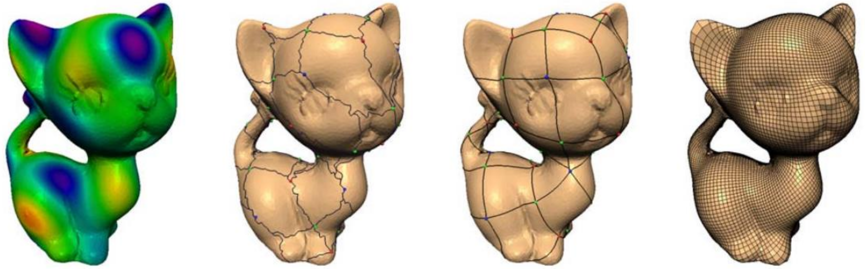
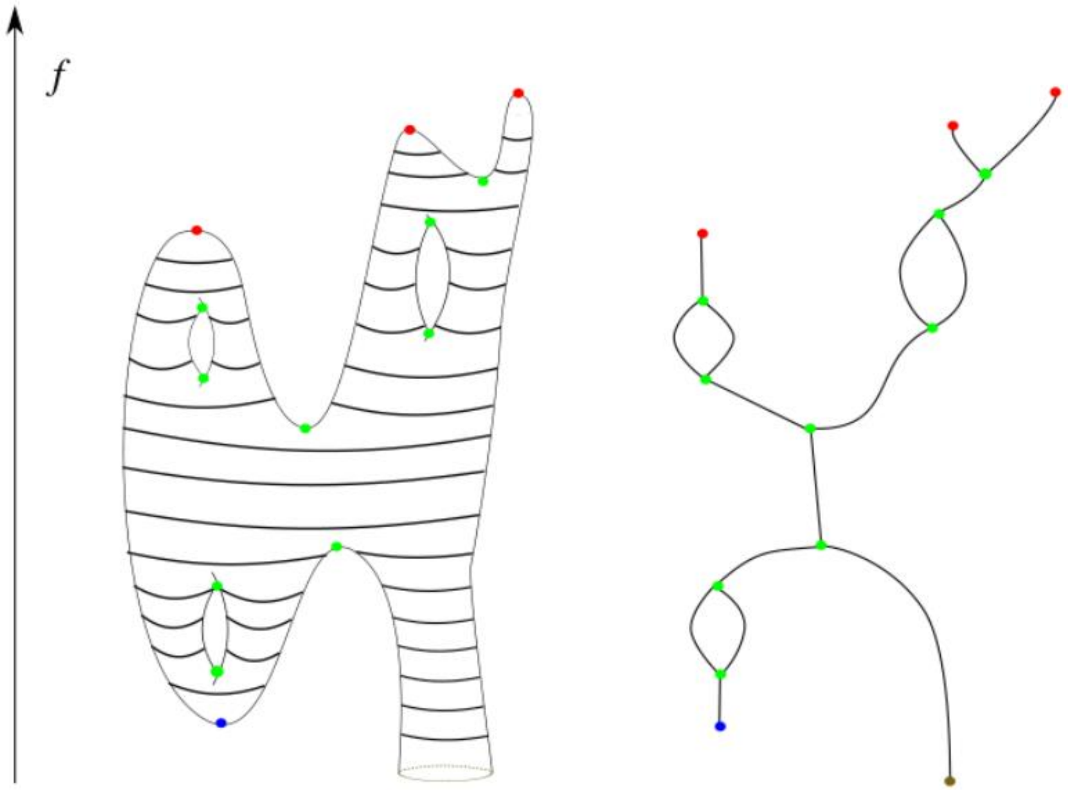


Cutting a surface into a disk (Ni et. el)

Reeb Graph

Given a surface M and a scalar function defined on it, we can define a combinatorial structure called the *Reeb graph* of M and f by collapsing the level sets of f as illustrated in the figure



Thank You