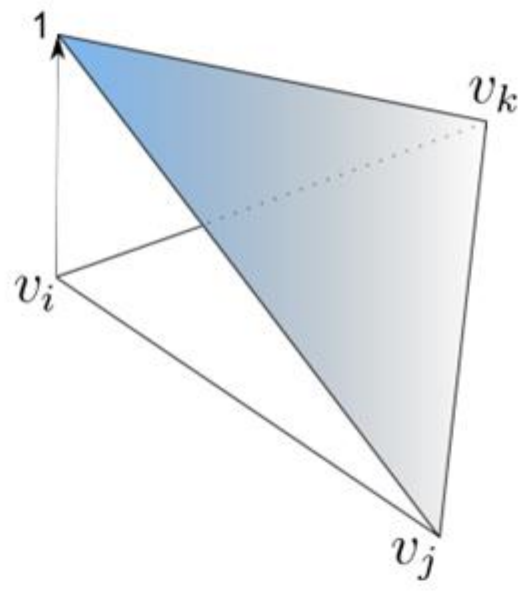
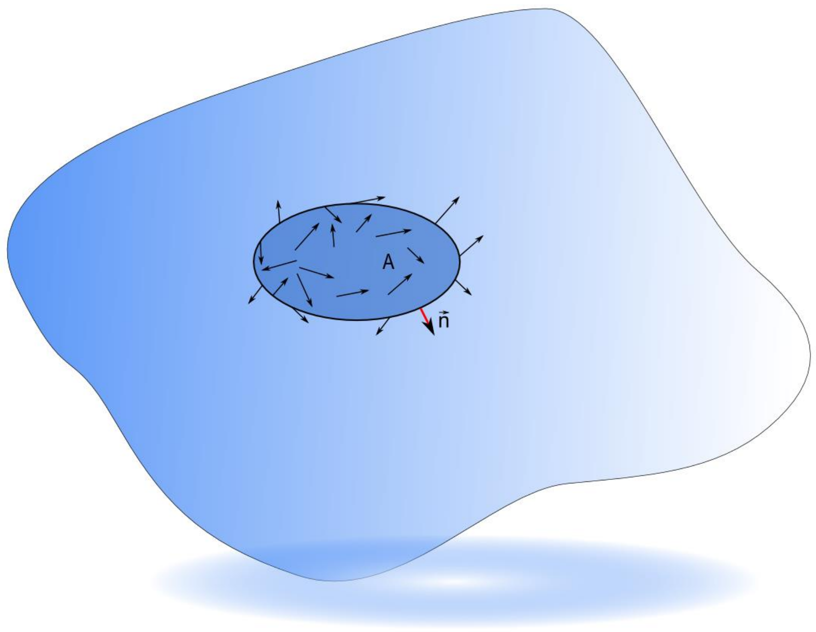


Discrete Operators on Triangulated Meshes



Mustafa Hajij

Piecewise linear functions on a triangulated meshes

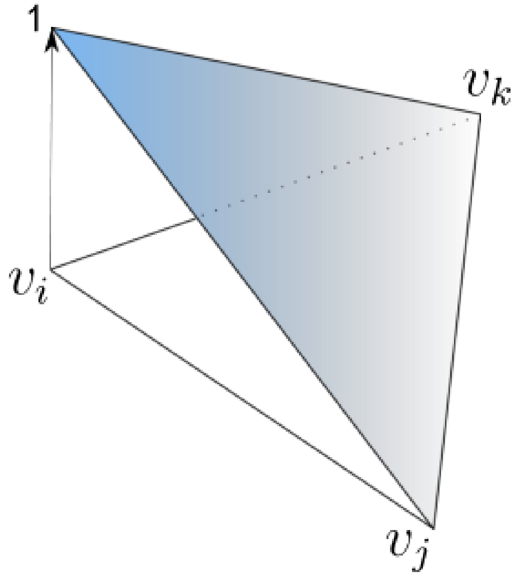
Suppose that the set vertex of Σ *is* $\{v_1, ..., v_n\}$ *and let* $f : \Sigma \longrightarrow \mathbb{R}$ *be a map defined on the vertices of* Σ. *Let* x *be a point in* $|\Sigma|$. *Then* $x \in |F|$ *where* $F = [v_i, v_j, v_k]$ *is a face in* Σ. *Hence, there are positive real numbers* λ_i, λ_j *and* λ_k *such that* $\lambda_i + \lambda_j + \lambda_k = 1$ *and* $x = \lambda_i v_i + \lambda_j v_j + \lambda_k v_k$. *Without loss of generality we can assume that* $i = 1, j = 2$ *and* $k = 3$. *Define the hat function* $B_i(x) : |\Sigma| \longrightarrow \mathbb{R}$ *by* $B_i(x) = \lambda_i$ *for* $i = 1, 2$ *and* 3 *and* $B_i(x) = 0$ *for* $i \geq 4$. *In particular* $B_i(v_j) = \delta_{ij}$ *for* $1 \leq i, j \leq 3$. *The extension of* f *can be written as :*

$$\hat{f}(x) = \sum_{i=1}^{n} f(v_i) B_i(x),$$

where $\sum_{i=0}^{n} B_i(x) = 1$ *and* $B_i(x) \geq 0$.

The hat function Bi

This function is called the hat function because if one defines B_i *on a face* $[v_i, v_j, v_k]$ *then the graph of the function of* B_i *looks like*



Gradient of a scalar function of a face in a mesh

Lemma *Let $f : F \longrightarrow \mathbb{R}$ be a piece-wise linear function defined on a face $F = [v_1, v_2, v_3]$ via the values $f(v_i)$, $i = 1, 2, 3$. Let E_i be the counterclockwise oriented edge opposite to the vertex v_i. Then the gradient of f is a constant and tangential vector on F given by :*

$$\text{grad } f = \nabla f = \frac{1}{2A_F} \sum_{i=1}^{3} f(v_i) ||E_i|| \overrightarrow{u_i} \tag{1.1}$$

where $\overrightarrow{u_i}$ is is a unit vector perpendicular to the edge E_i and oriented so that it

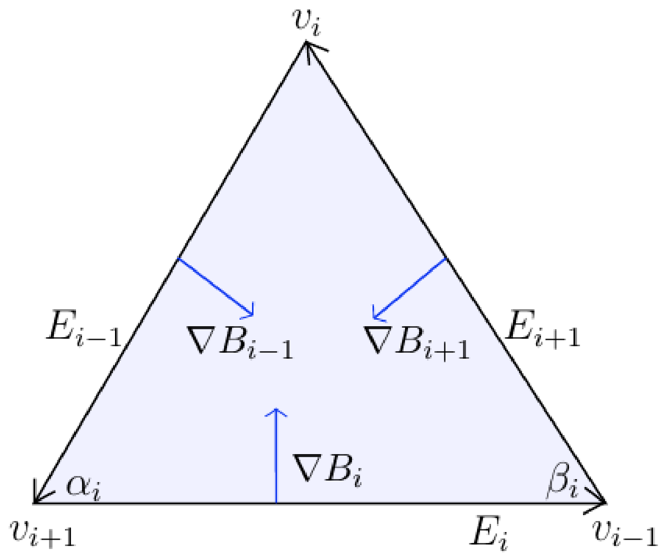


FIGURE 1.1. The gradients of hat functions of triangle.

points into the face F. Moreover, $\overrightarrow{u_i} = \frac{1}{||E_i||}(E_{i+1} \cot \alpha_i - E_{i-1} \cot \beta_i)$ where α_i and β_i are the angles on both sides of the edge E_i, α_i is the angle opposite the edge E_{i+1} and β_i is the angle opposite to the edge E_{i-1}.

Gradient of a scalar function of a face in a mesh

Proof. Let r be an arbitrary point inside the triangle F. Then r can be written as

$$r = B_1(r)v_1 + B_2(r)v_2 + B_3(r)v_3.$$

where $B_i : F \longrightarrow \mathbb{R}$ is the hat function on the vertex v_i defined by $B_i(v_j) = \delta_{ij}$ for $i, j = 1, 2, 3$. Hence any the function f has the form:

$$f(r) = \sum_{i=1}^{3} f(v_i)B_i(r) \tag{1.2}$$

Now the barycentric coordinate B_i is given by the ratio $B_i(r) = \frac{A_i(r)}{A_F}$ where $A_i = \frac{1}{2}h_i||E_i||$. See Figure 1.2. Since A_F and $||E_i||$ are invariant for the same face F,

Gradient of a scalar function of a face in a mesh

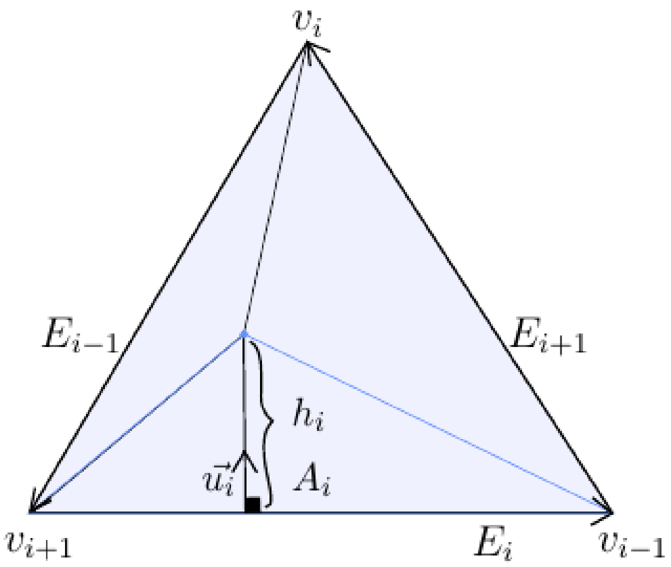


FIGURE Barycentric coordinates.

then B_i is a function of h_i. Hence

$$\nabla B_i = \frac{d}{dh_i}\Big(\frac{h_i||E_i||}{2A_F}\Big)\overrightarrow{u_i} = \frac{||E_i||}{2A_F}\overrightarrow{u_i} \tag{1.3}$$

where $\overrightarrow{u_i}$ is a unit vector perpendicular to the vector E_i and oriented so that it points into the face F. Now equation (1.2) implies :

$$\nabla f = \sum_{i=1}^{3} \nabla B_i f(v_i),$$

Hence (1.1) follows. For the second part of the Lemma, choosing $\overrightarrow{u_i} = \frac{E_{i-1}\times E_i}{||E_{i-1}\times E_i||} \times \frac{E_i}{||E_i||}$, equation (1.3) becomes:

Gradient of a scalar function of a face in a mesh

$$\nabla B_i = \frac{1}{2A_F} \frac{E_{i-1} \times E_i}{||E_{i-1} \times E_i||} \times E_i$$

However,

$$\begin{aligned}
\frac{E_{i-1} \times E_i}{||E_{i-1} \times E_i||} \times E_i &= \frac{-\langle E_i, E_i \rangle E_{i-1} + \langle E_i, E_{i-1} \rangle E_i}{||E_{i-1} \times E_i||} \\
&= \frac{-\langle E_i, E_i \rangle E_{i-1} - \langle E_i, E_{i-1} \rangle E_{i-1} + \langle E_i, E_{i-1} \rangle E_{i-1} + \langle E_i, E_{i-1} \rangle E_i}{||E_{i-1} \times E_i||} \\
&= \frac{\langle E_i, E_{i+1} \rangle E_{i-1} + \langle E_i, E_{i-1} \rangle (-E_{i+1})}{||E_{i-1} \times E_i||}
\end{aligned}$$

where in the last equation we used the fact $E_{i-1} + E_i + E_{i+1} = 0$. One the other

$$||E_{i-1} \times E_i|| = ||E_i \times E_{i+1}|| = 2A_F$$

Hence,

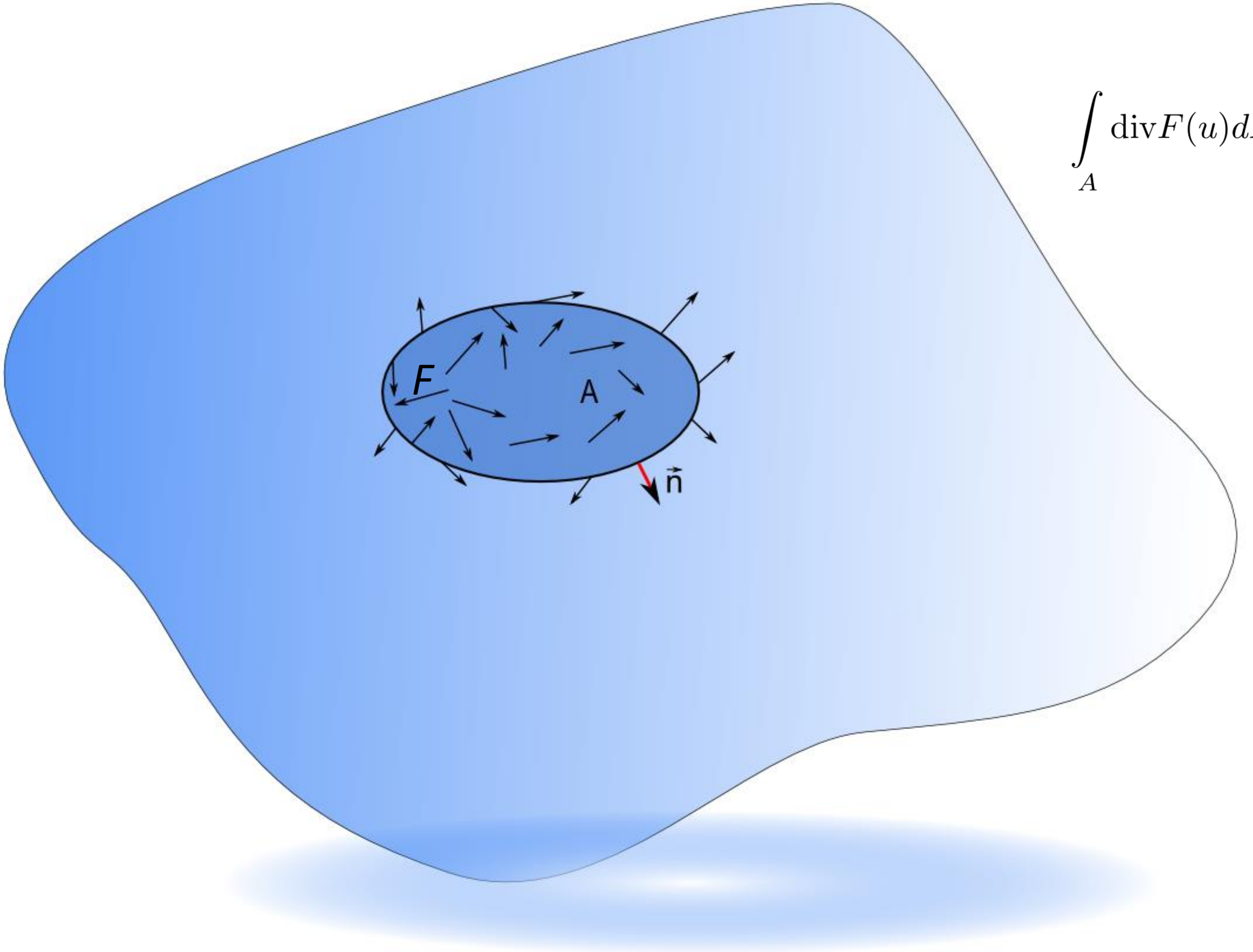
$$\begin{aligned}
\frac{\langle E_i, E_{i+1} \rangle E_{i-1} - \langle E_i, E_{i-1} \rangle E_{i+1}}{||E_{i-1} \times E_i||} &= \frac{\langle E_i, E_{i+1} \rangle E_{i-1}}{||E_i \times E_{i+1}||} - \frac{\langle E_i, E_{i-1} \rangle E_{i-1}}{||E_{i-1} \times E_i||} \\
&= \frac{\langle -E_i, E_{i+1} \rangle (-E_{i-1})}{||-E_i \times E_{i+1}||} + \frac{\langle E_i, -E_{i-1} \rangle E_{i+1}}{||-E_{i-1} \times E_i||}
\end{aligned}$$

However, since the cotangent of an angel between two vectors v and w is equal to $\frac{\langle v,w \rangle}{||v \times w||}$, then

$$\frac{E_{i-1} \times E_i}{||E_{i-1} \times E_i||} \times E_i = (\cot \alpha) E_{i+1} - (\cot \beta) E_{i-1}.$$

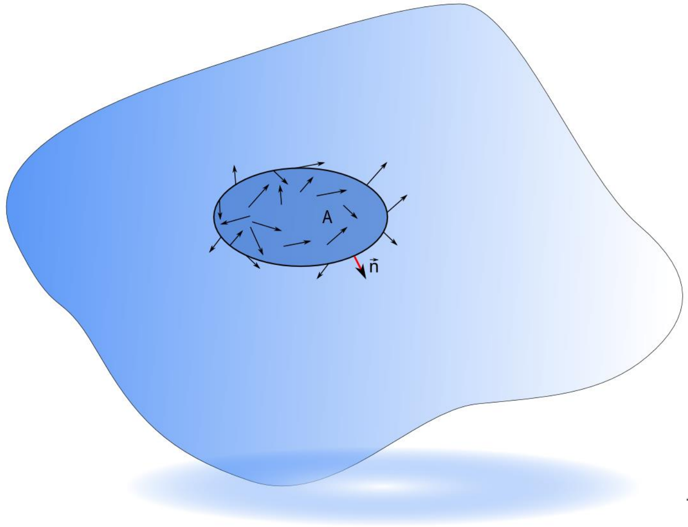
The result follows. □

The divergence theorem



$$\int_A \mathrm{div} F(u) dA = \int_{\partial A} < F(u), n(u) > ds$$

Discrete Laplace-Beltrami operator



$$\int_A \mathrm{div}F(u)dA = \int_{\partial A} < F(u), n(u) > ds$$

In particular, if $f : M \to \mathbb{R}$ is a smooth scalar field on the surface M then we can write $\mathrm{div}\nabla f = \Delta f$ and hence

$$\int_A \Delta f A = \int_A \mathrm{div}\nabla f dA = \int_{\partial A} < \nabla f, n(u) > ds$$

when A is very small

$$\Delta f \approx \frac{1}{A} \int_{\partial A} < \nabla f, n(u) > ds$$

Discrete Laplace-Beltrami operator

On a triangulated mesh around a vertex v_i we have

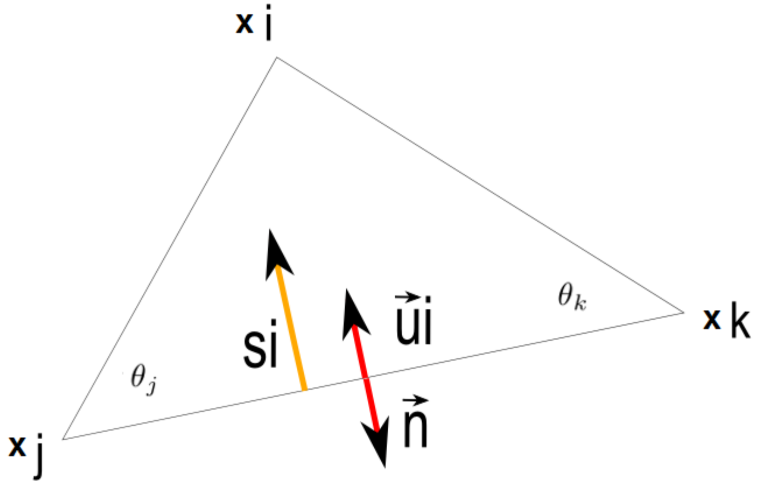
$$\Delta f(v_i) \approx \frac{1}{A} \sum_{all\ \mathrm{faces\ T_q\ round\ v_i}} \int_{\partial A \cap T_q} < \nabla f, \vec{n}_q > ds$$

where n_q is the normal the edge in the face T_q opposite to the vertex v_i.
Hence, we must determine :

$$\int_{\partial A \cap T} < \nabla f, \vec{n} > ds$$

Since $< \nabla f, \vec{n} >$ is a constant on T then

$$\begin{aligned} \int_{\partial A \cap T} &< \nabla f, \vec{n} > ds =< \nabla f, \vec{n} > \int_{\partial A \cap T} ds \\ &= < \nabla f, \vec{n} > ||x_j - x_k|| \\ &= < \nabla f, -||x_j - x_k|| \vec{u} > \\ &= < \nabla f, -s_i > \end{aligned}$$



where $||x_j - x_k|| \vec{u} = s_i$

Discrete Laplace-Beltrami operator

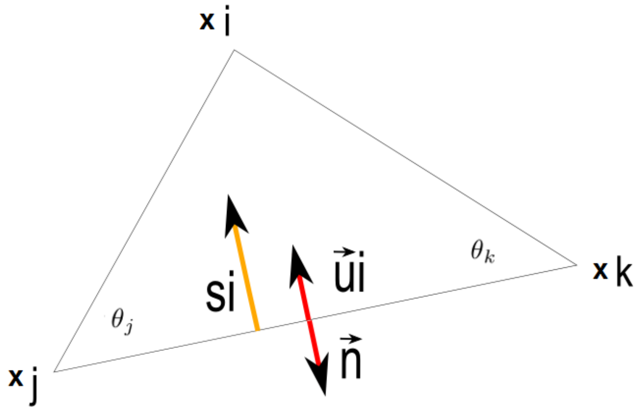
On the other hand one has at any point x in the triangle $T = [x_i, x_j, x_k]$

$$\nabla f(x) = f_i \nabla B_i(x) + f_j \nabla B_j(x) + f_k \nabla B_k(x)$$

since $B_i(x) + B_j(x) + B_k(x) = 1$ then $\nabla B_i(x) + \nabla B_j(x) + \nabla B_k(x) = 0$.
Hence at any point x in the triangle $T = [x_i, x_j, x_k]$ we have

$$\begin{aligned}
\nabla f(x) &= f_i(-\nabla B_j(x) - \nabla B_k(x)) + f_j \nabla B_j(x) + f_k \nabla B_k(x) \\
&= (f_j - f_i)\nabla B_j(x) + (f_k - f_i)\nabla B_k(x) \\
&= (f_j - f_i)\frac{1}{2A_T}||x_j - x_i||\vec{u}_j + (f_k - f_i)\frac{1}{2A_T}||x_k - x_i||\vec{u}_k \\
&= \frac{1}{2A_T}((f_j - f_i)s_j + (f_k - f_i)s_k)
\end{aligned}$$

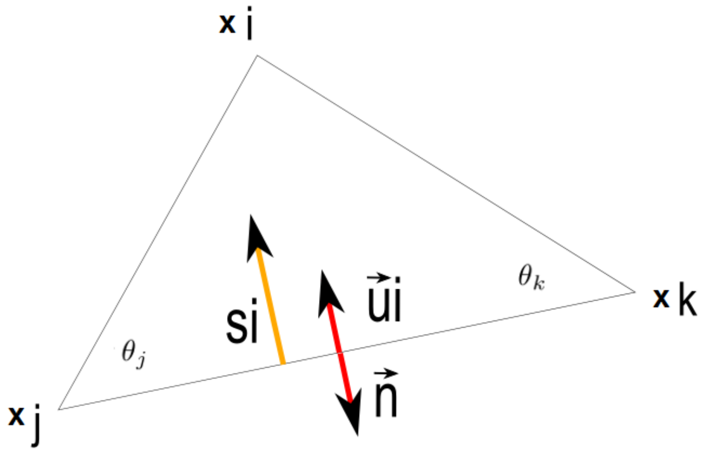
Hence



Discrete Laplace-Beltrami operator

Hence

$$< -s_i, \nabla f(x) >= \frac{1}{2A_T}((f_j - f_i) < -s_i, s_j > +(f_k - f_i) < -s_i, s_k >$$



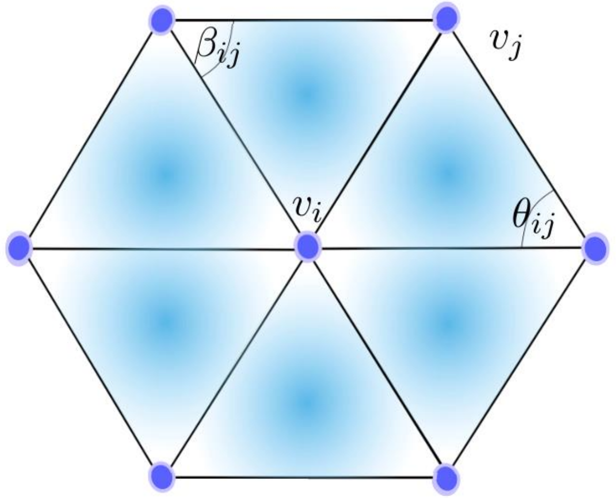
However, $< -s_i, s_j >= -||x_j - x_k||||x_k - x_i|| \cos\theta_k$ and $A_T = \frac{1}{2}\ ||x_j - x_k||||x_k - x_i|| \sin\theta_k$ and similar formulas can be written for $< -s_i, s_k >$

Hence

$$< -s_i, \nabla f(x) >= ((f_j - f_i) \cot\theta_k + (f_k - f_i) \cot\theta_j)$$

When we sum on every triangle around the vertex v_i, every edge $[v_i, v_j]$ contributes by two summands $(f_j - f_i)\cot\theta_{i,j}$ and $(f_j - f_i)\cot\beta_{i,j}$. Namely, the edge $[v_i, v_j]$ contributes by $(\cot\theta_{i,j} + \cot\beta_{i,j})(f_j - f_i)$ hence

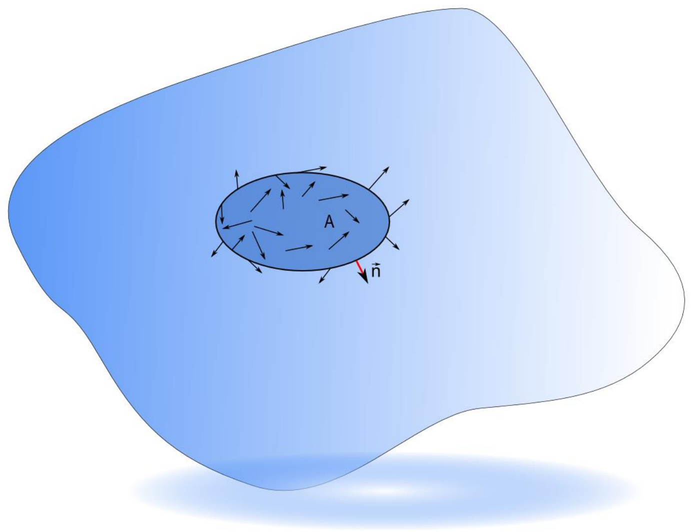
$$\Delta f(v_i) = \frac{1}{A_i} \sum_{v_j \in lk(v_i)} (\cot\theta_{i,j} + \cot\beta_{i,j})(f_j - f_i)$$



Discrete divergence on a triangulated meshes

If $A \subset M$ and X is a vector field on a smooth surface a smooth surface divergence theorem states :

$$\int_A divX(v)dA = \int_{\partial A} < X(v), n(v) > ds$$

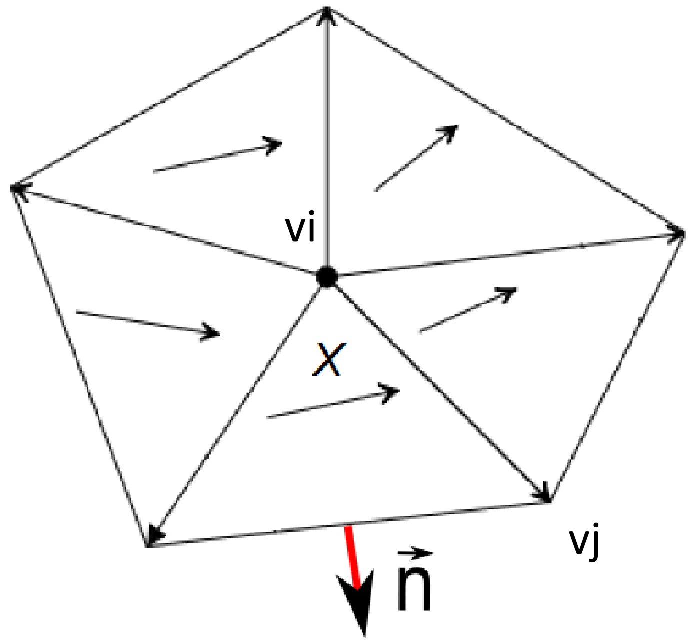


This suggests the following discretization on a triangulated mesh.

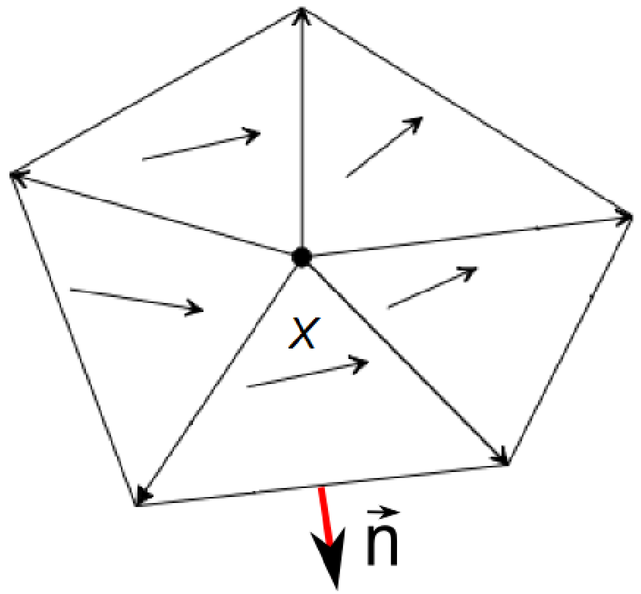
Suppose that $X : F(M) \to \mathbb{R}^3$ is a vector field that assigns to every face F in M a vector $X(F)$ that lies completely in F, then the divergence of X at a vertex v_i is defined as

$$divX(u_i) = c \int_{j \in \text{face around } (u_i)} < X_j, n_j > ds$$

where n_j is the normal the edge in the face F_j opposite to the vertex v_i and c is a constant that depends on the neighborhood of v_i. Some calculations implies the explicit formula :



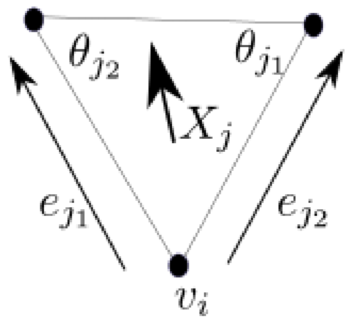
Discrete divergence on a triangulated meshes



some calculations implies the explicit formula :

$$\operatorname{div} X(v_i) = \frac{1}{2} \sum_{j \in F(i)} \cot \theta_{j_1} \langle e_{j_1}, X_j \rangle + \cot \theta_{j_2} \langle e_{j_2}, X_j \rangle \,,$$

where $F(i)$ is the set of indices of all faces that are incident to the vertex v_i, e_{j_1}, e_{j_2} are the two vectors in face j that contain the vertex v_i and $\theta_{j_1}, \theta_{j_2}$ are the angles that are opposite the edges e_{j_1} and e_{j_2} respectively. See Figure 4.3.



The discrete operators are consistent

One can show that

$$\Delta f = \mathrm{div}\ \nabla f$$