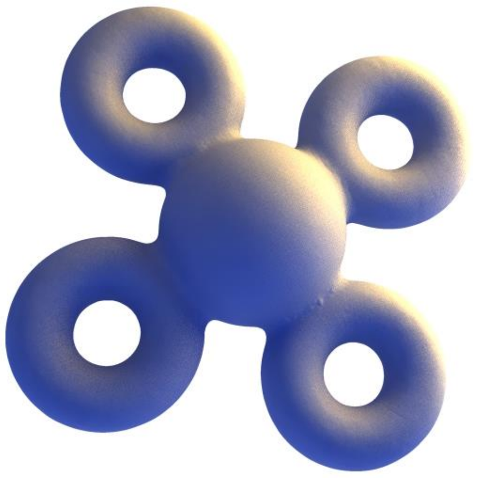
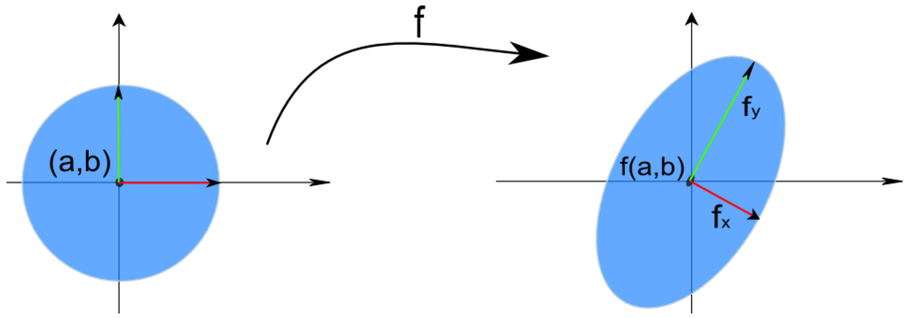
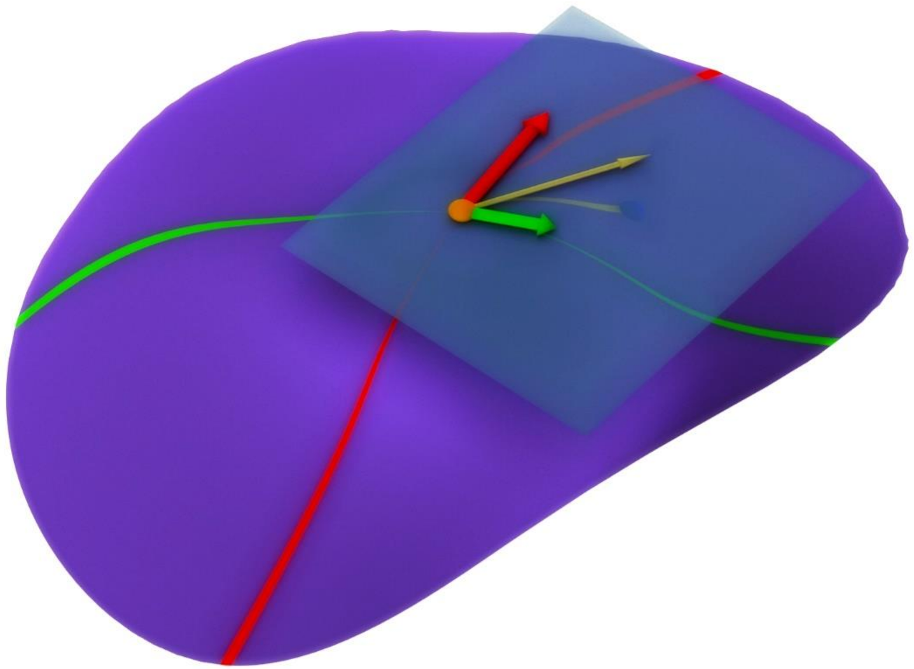




Surface Parametrization



Mustafa Hajij

Parametrized surfaces

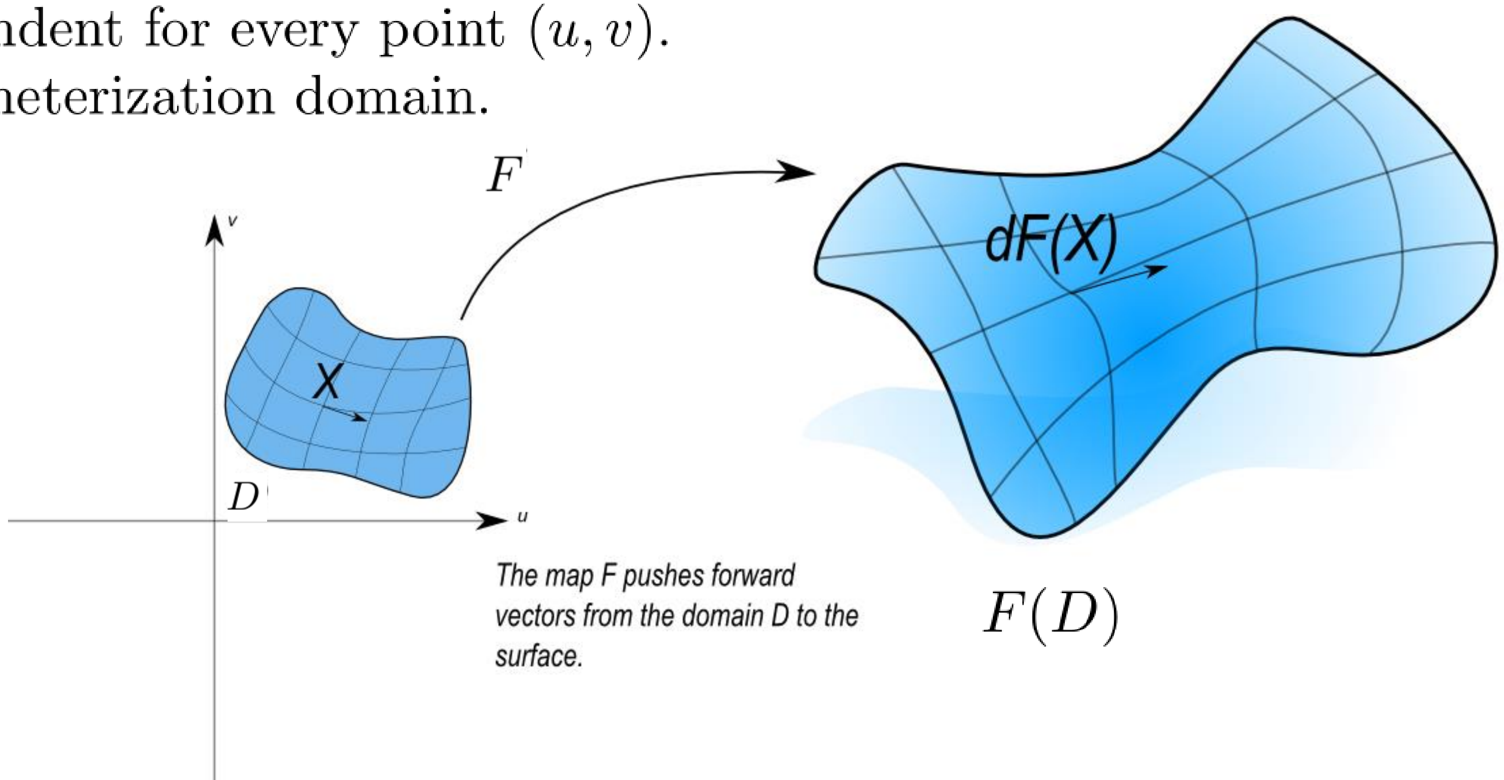
Let $f : D \to f(D)$ be a smooth that maps a domain $D \subset \mathbb{R}^2$ to the surface $f(D) \subset \mathbb{R}^3$.

In other words,

$$f(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

such a map is called a *regular parameterization* for the surface $f(D)$ if the vectors f_u and f_v are linearly independent for every point (u,v).

The domain D is called the parameterization domain.



The map F pushes forward vectors from the domain D to the surface.

$F(D)$

Click below to see an example for a parametrization map

http://mathinsight.org/parametrized_surface_introduction

Taylor Expansion Reminder

Let $f\colon D \subset \mathbb{R} \to \mathbb{R}$ be an infinitly smooth function in some domain D. Let $a \in D$. The single variable Taylor expansion of f around a is given by

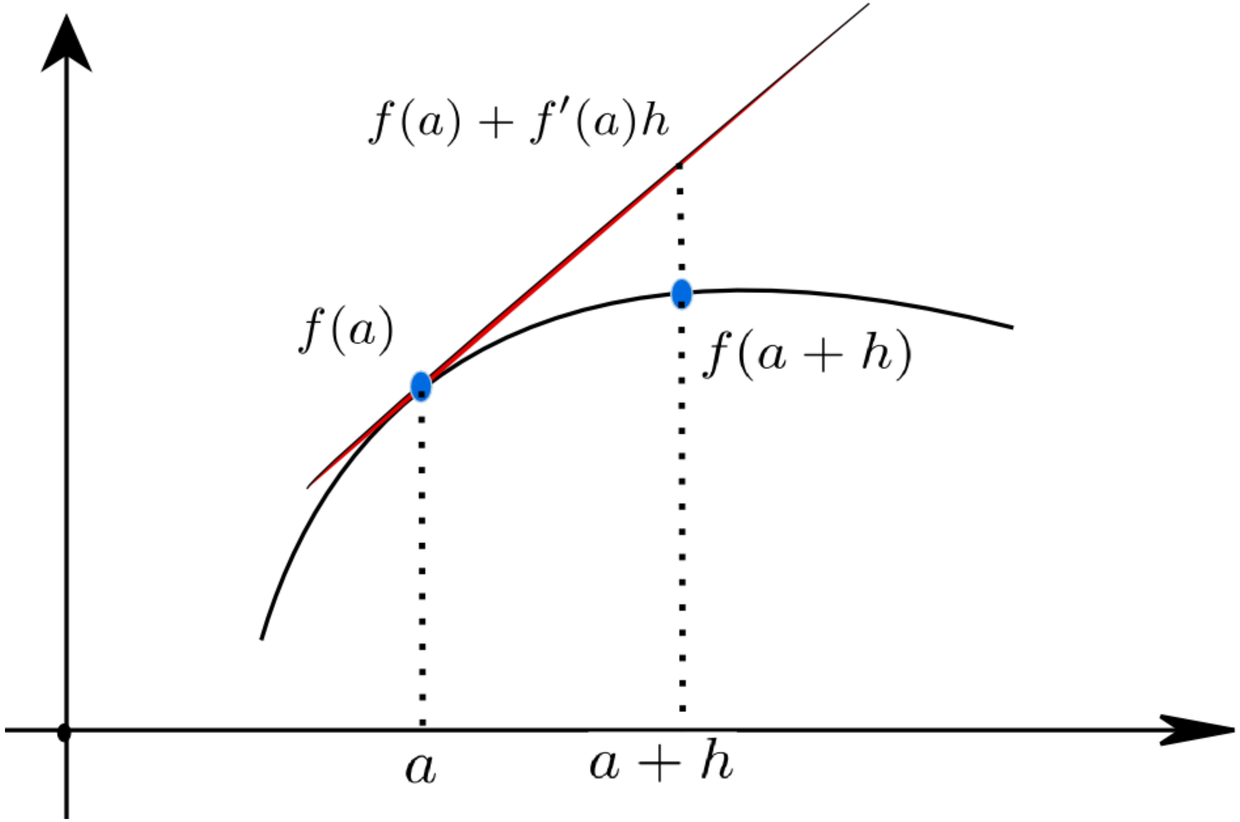
$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \ldots$$

We can approximate f linearly around a :

$$f(x) \simeq f(a) + f'(a)(x-a)$$

Write $h = x - a$, then

$$f(a+h) \simeq f(a) + f'(a)h$$



Taylor Expansion Reminder

Let $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}^2$ be an infinitly smooth function in some domain D. Let $(a,b) \in D$. The two variables Taylor expansion of f around (a,b) is given by

$$f(u,v) = f(a,b) + f_u(a,b)(u-a) + f_v(a,b)(v-b) + \dots$$

write $(h,k) = (u,v) - (a,b)$, then $f(a+h, b+k)$ can be approximated by

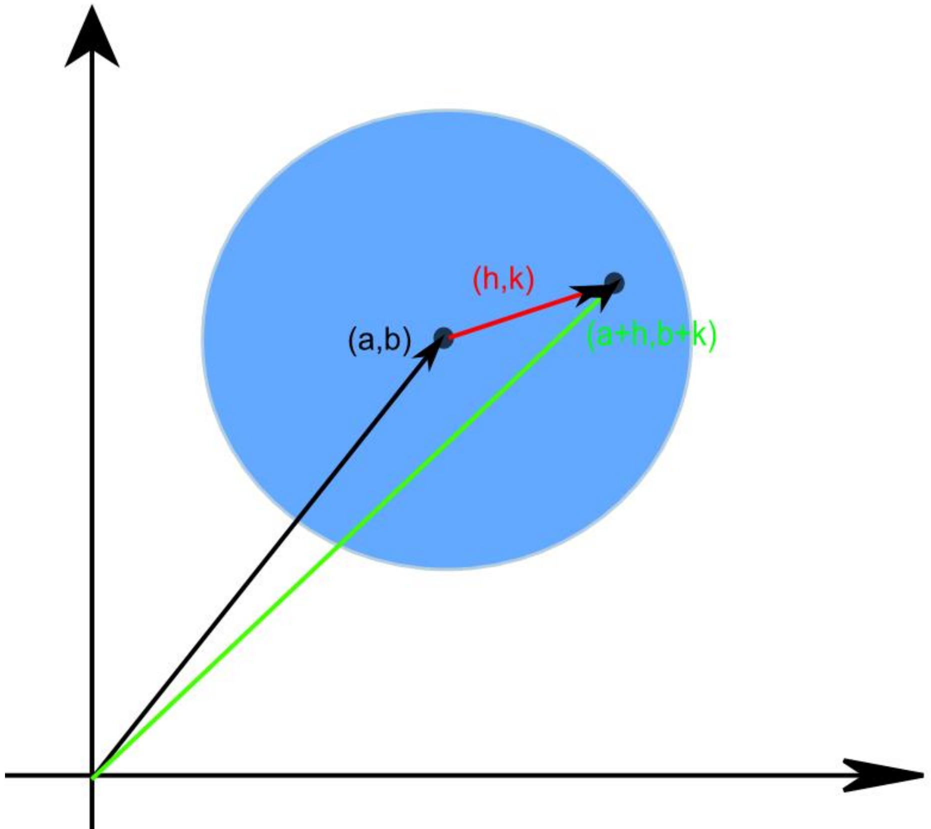
$$f(a+h, b+k) \simeq f(a,b) + f_u(a,b)h + f_v(a,b)k$$

or

$$f(a+h, b+k) \simeq f(a,b) + \left[\begin{array}{cc} f_u(a,b) & f_v(a,b) \end{array}\right] \left[\begin{array}{c} h \\ k \end{array}\right]$$

Define $J := \left[\begin{array}{cc} f_u(a,b) & f_v(a,b) \end{array}\right]$

J is called the Jacobian



The Jacobian

Similarly,

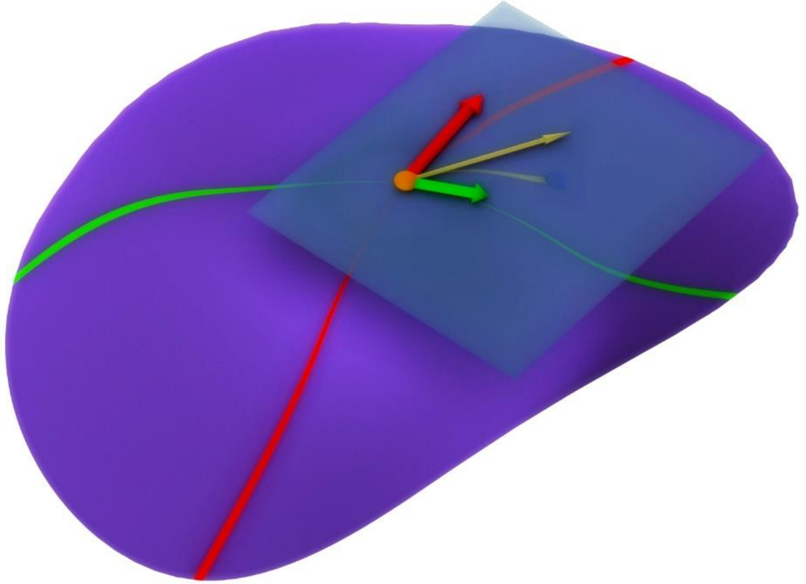
Let $f{:}D \to F(D)$ be a regular parameterization. Let $(a,b) \in D$. Then, similar to the argument above

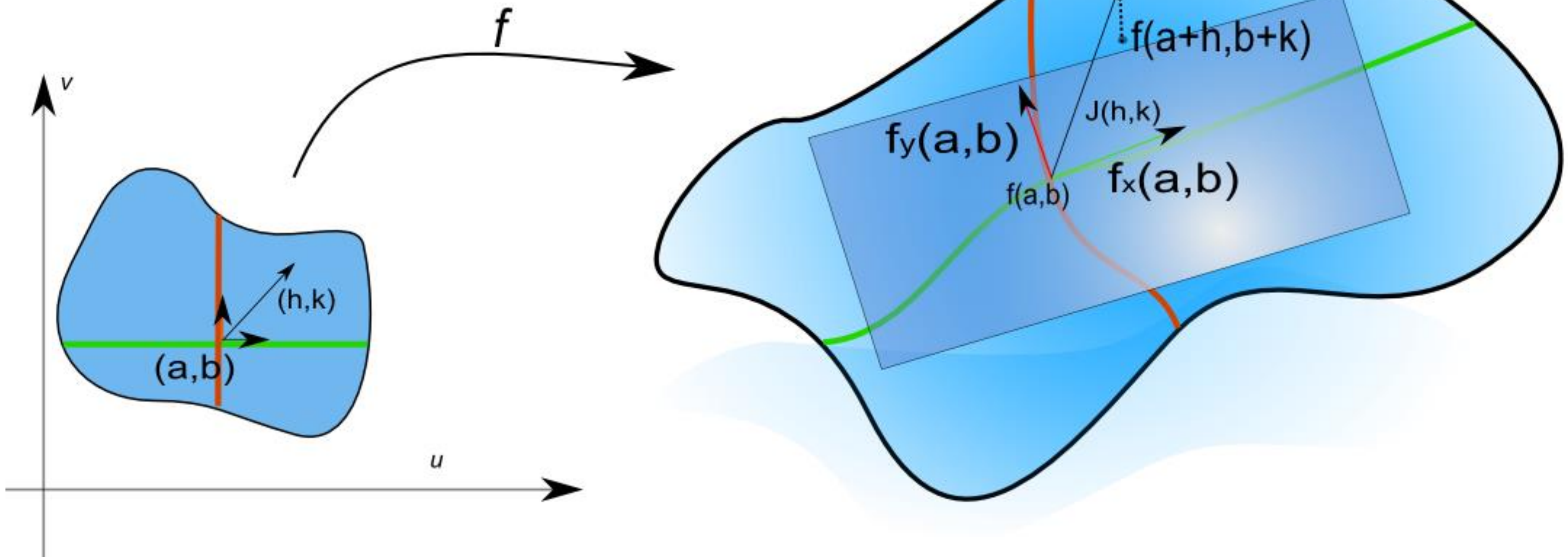
$$f(a+h, b+k) \simeq f(a,b) + J \begin{bmatrix} h \\ k \end{bmatrix}$$

where

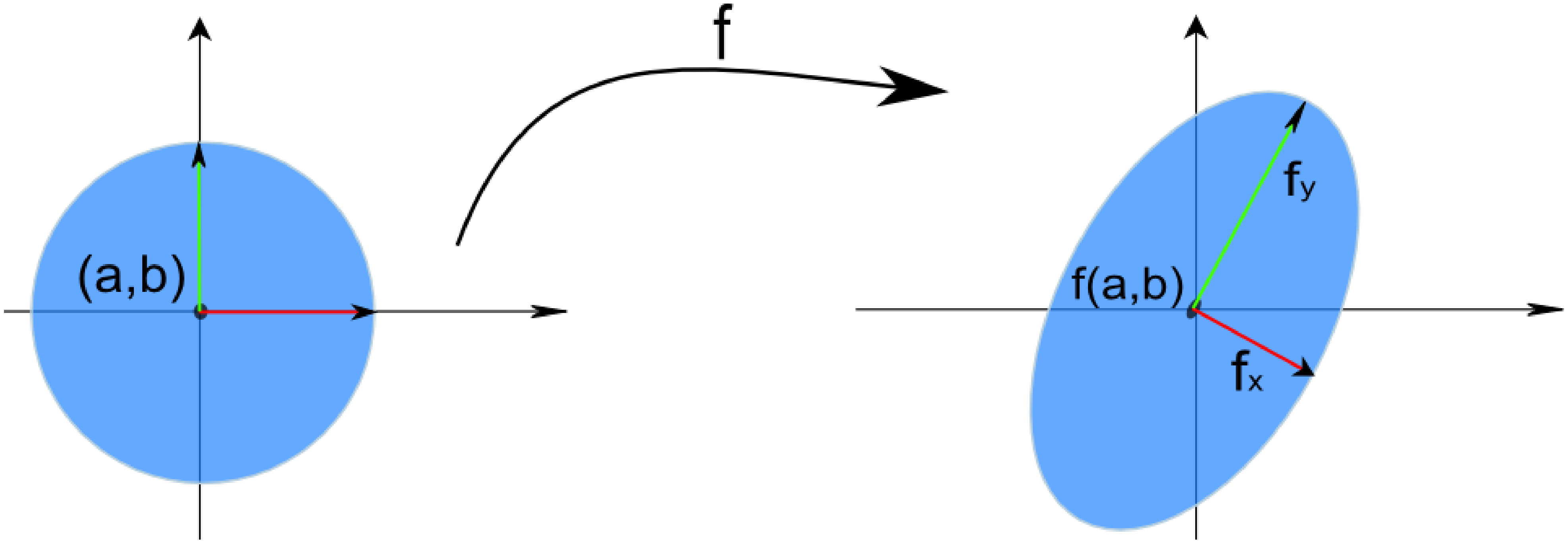
$$J = \begin{bmatrix} f_u(a,b) & f_v(a,b) \end{bmatrix}$$

$$\begin{bmatrix} f_u(a,b) & f_v(a,b) \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$$





The Jacobian and the anisotropy ellipse



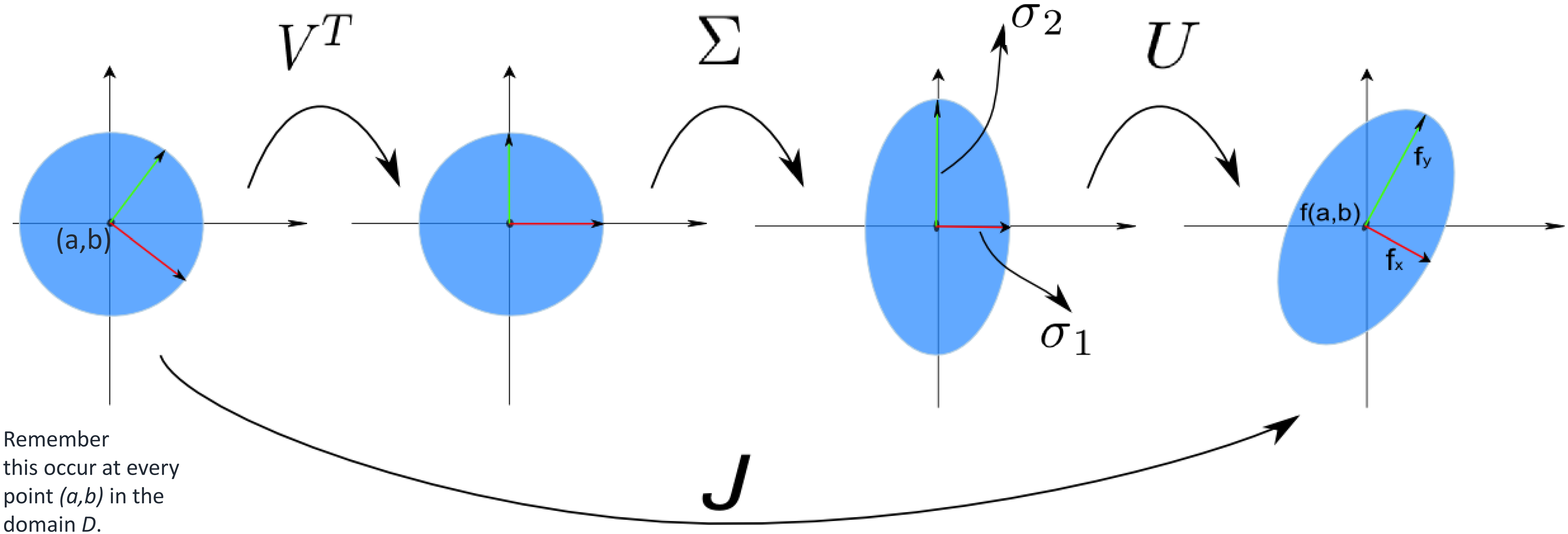
The anisotropy ellipse

The singular value decomposition of the Jacobian

The singular value decomposition of the Jacobian

$$J = U\Sigma V^T = U \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} V^T$$

with singular values $\sigma_1 \geq \sigma_2 > 0$ and orthonormal matrices $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{2\times 2}$



Remember this occur at every point *(a,b)* in the domain *D*.

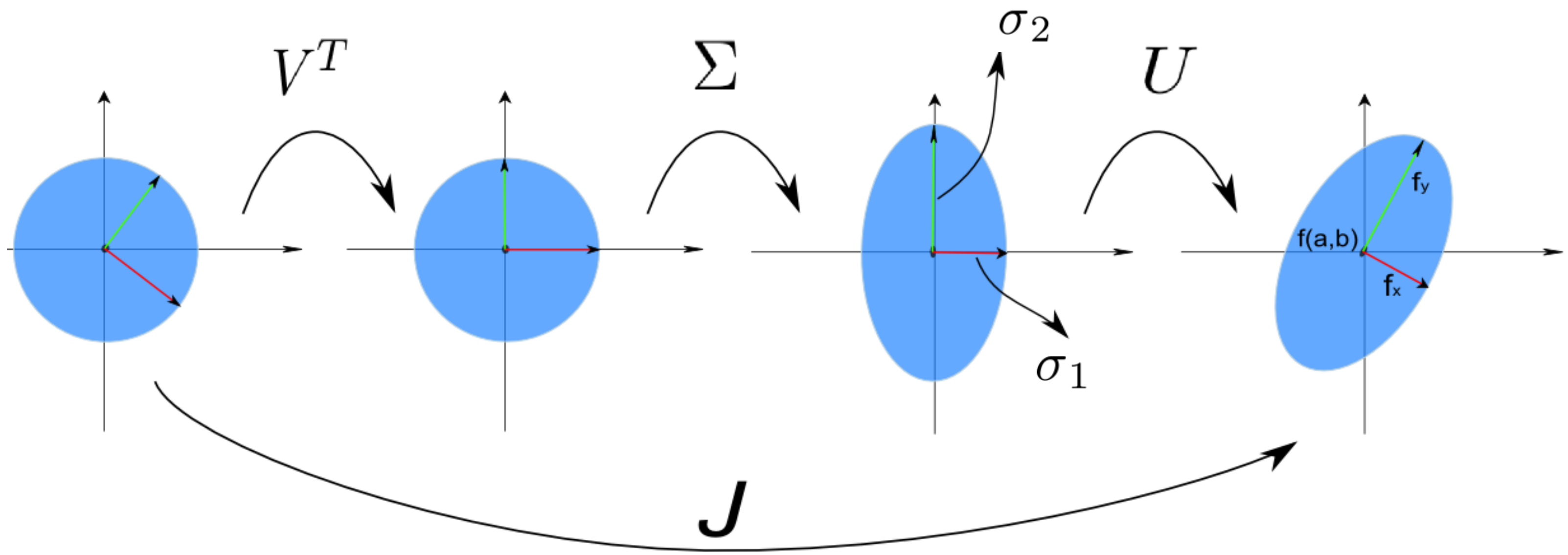
Structure-preserving parametrization

The singular value decomposition of the Jacobian

$$J = V^T \Sigma U = V^T \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} U$$

We care about three types of parametrization :

f is an isometry (length-preserving) $\iff \sigma_1 = \sigma_2 = 1$
f is conformal (angle-preserving) $\iff \sigma_1 = \sigma_2$
f is equiareal (area-preserving) $\iff \sigma_1 \sigma_2 = 1$



The first fundamental form

Write $f_1 = f_u$, $f_2 = f_v$, and $g_{ij} =< f_i, f_j >$. The first fundamental form is as the symmetric matrix :

$$I = \begin{bmatrix} < f_1, f_1 > & < f_1, f_2 > \\ < f_2, f_1 > & < f_2, f_2 > \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

The singular values of I are $\lambda_1 = \sigma_1^2$ and $\lambda_2 = \sigma_2^2$.

The relation between the Jacobian and the first fundamental form :

$$\begin{aligned} I &= J^T J = (U\Sigma V^T)^T (U\Sigma V^T) \\ &= V\Sigma^T U^T U\Sigma V^T \\ &= V\Sigma^T \Sigma V^T \\ &= V \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} V^T \end{aligned}$$

The first fundamental form

Hence,

$$
\begin{aligned}
&f \text{ is an isometry (length-preserving)} \iff \sigma_1 = \sigma_2 = 1 \iff \lambda_1 = \lambda_2 = 1 \\
&f \text{ is conformal (angle-preserving)} \iff \sigma_1 = \sigma_2 \iff \lambda_1 = \lambda_2 \\
&f \text{ is equiareal (area-preserving)} \iff \sigma_1 \sigma_2 = 1 \iff \lambda_1 \lambda_2 = 1
\end{aligned}
$$

The lengths of the axes of the anisotropy ellipse correspond to the eigenvalues of the first fundamental form. Hence they can be found by computing the square roots of the zeros of the characteristic polynomial of the matrix *I* :

$$
\begin{aligned}
\sigma_1 &= \sqrt{1/2(E+G) + \sqrt{(E-G)^2 + 4F^2}} \\
\sigma_2 &= \sqrt{1/2(E+G) - \sqrt{(E-G)^2 + 4F^2}}
\end{aligned}
$$

Isometry

Isometry=conformal+equiareal

Isometry parametrization is the ideal one but it cannot always be achieved.

Conformal maps

Conformal maps preserves angles.

They also map small circles to circles. This is also called *circle packing*.