Topological Algorithms-II

Mustafa Hajij

Cutting Graph

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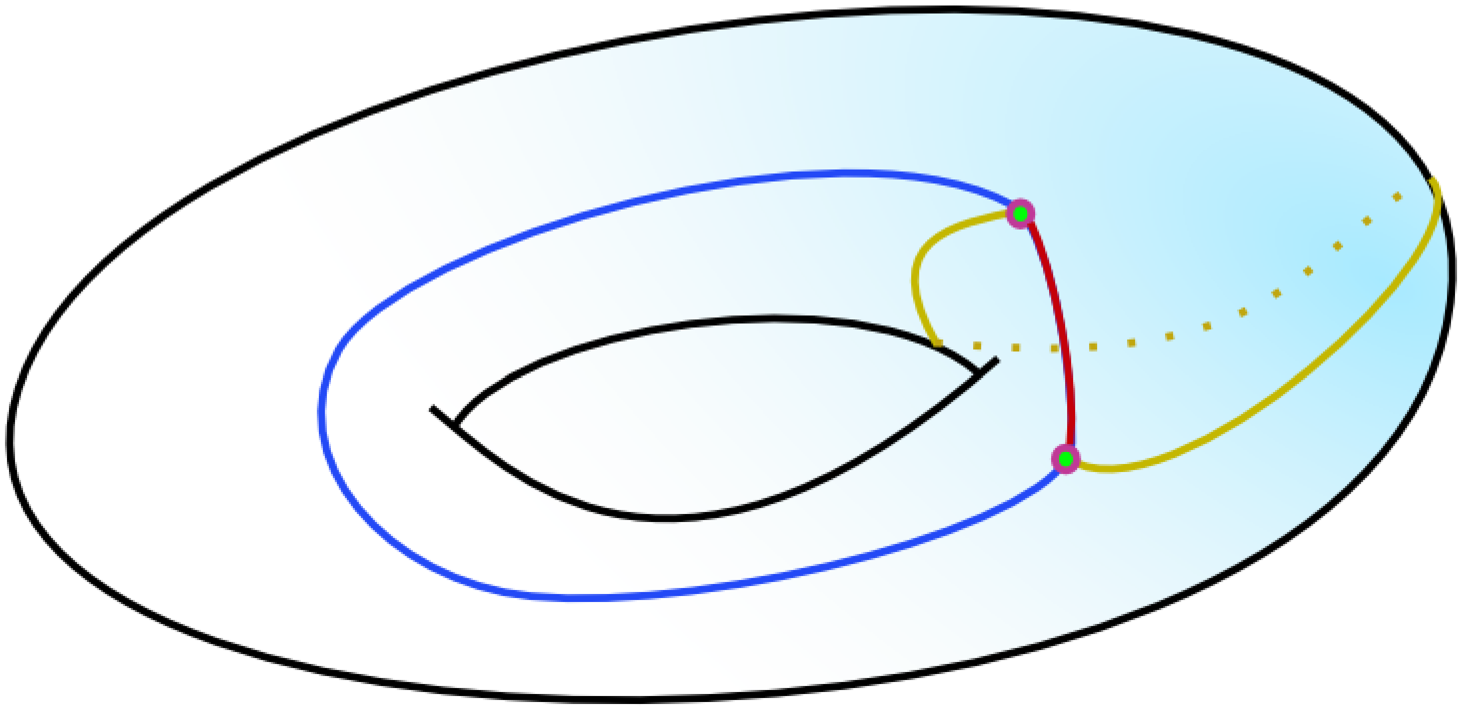

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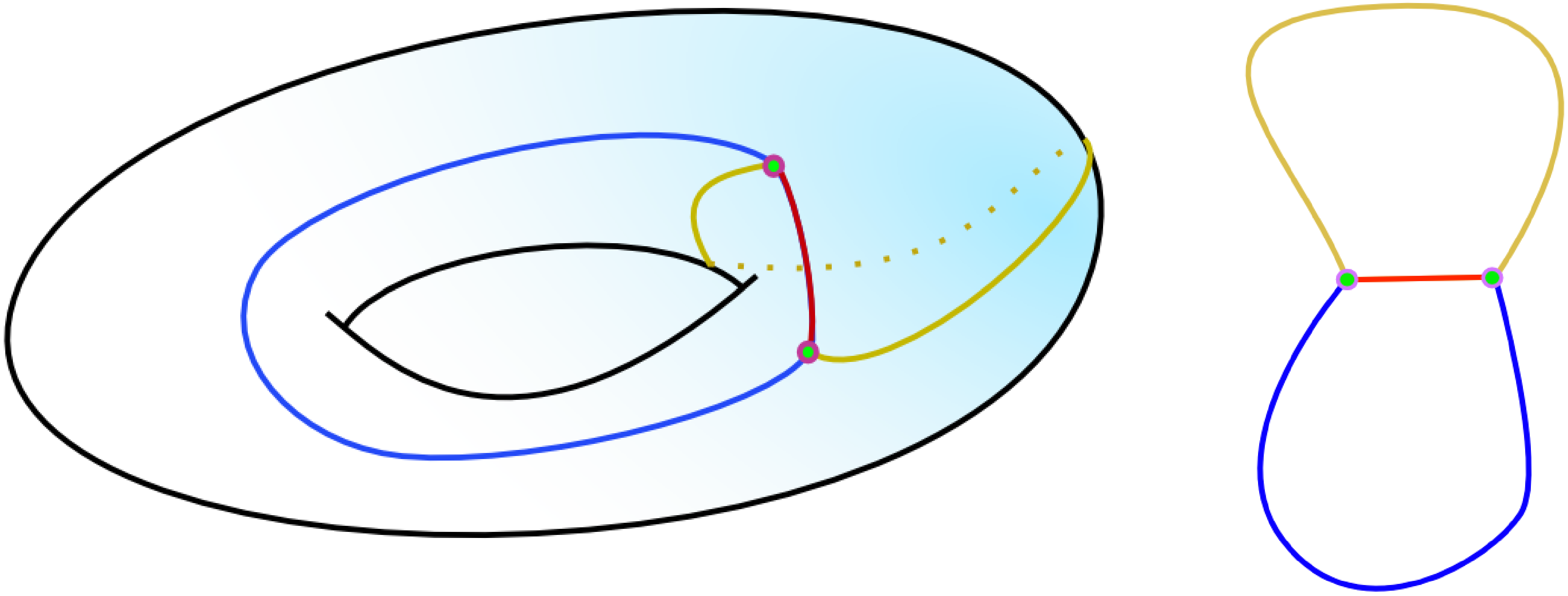

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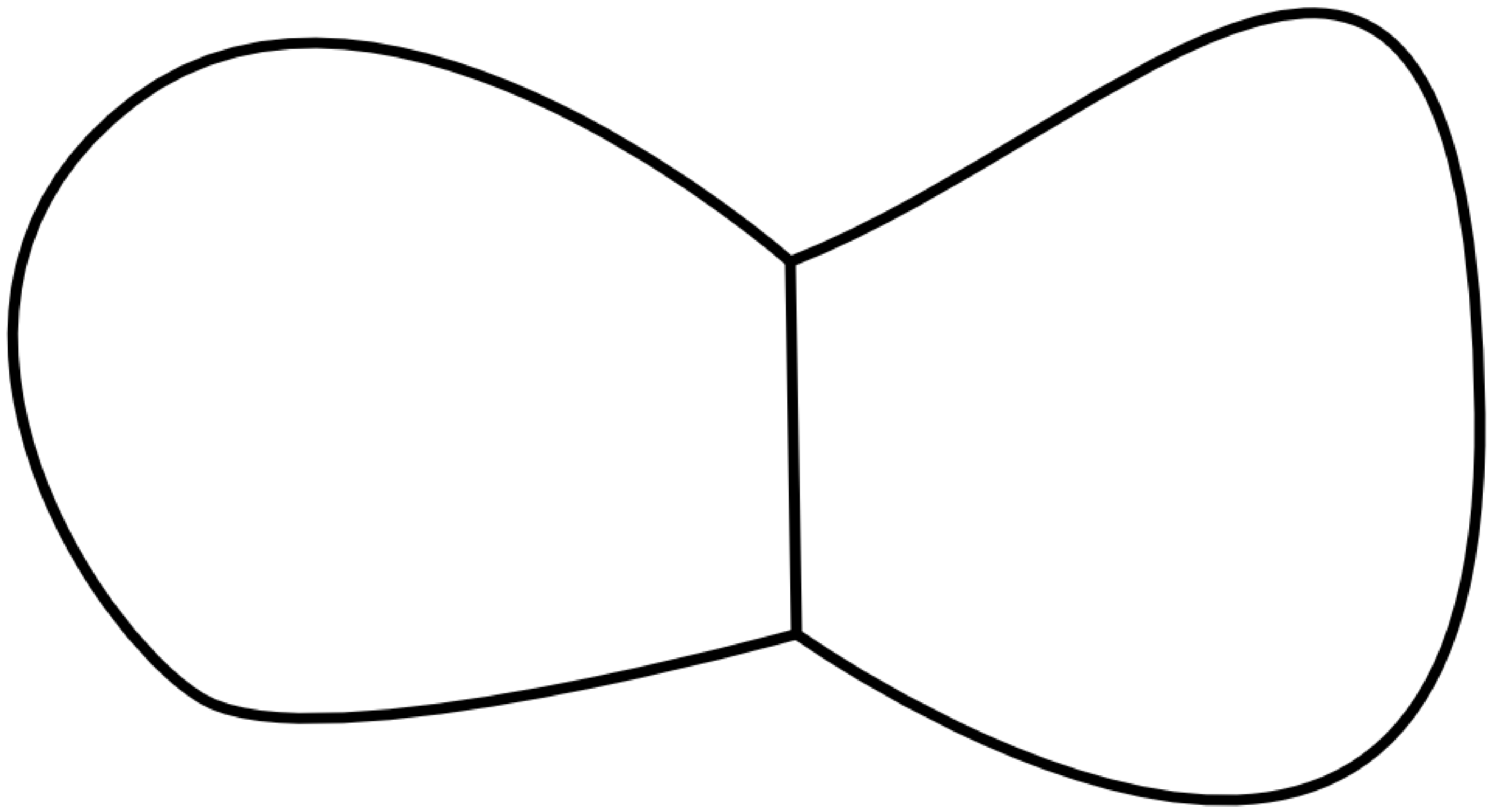
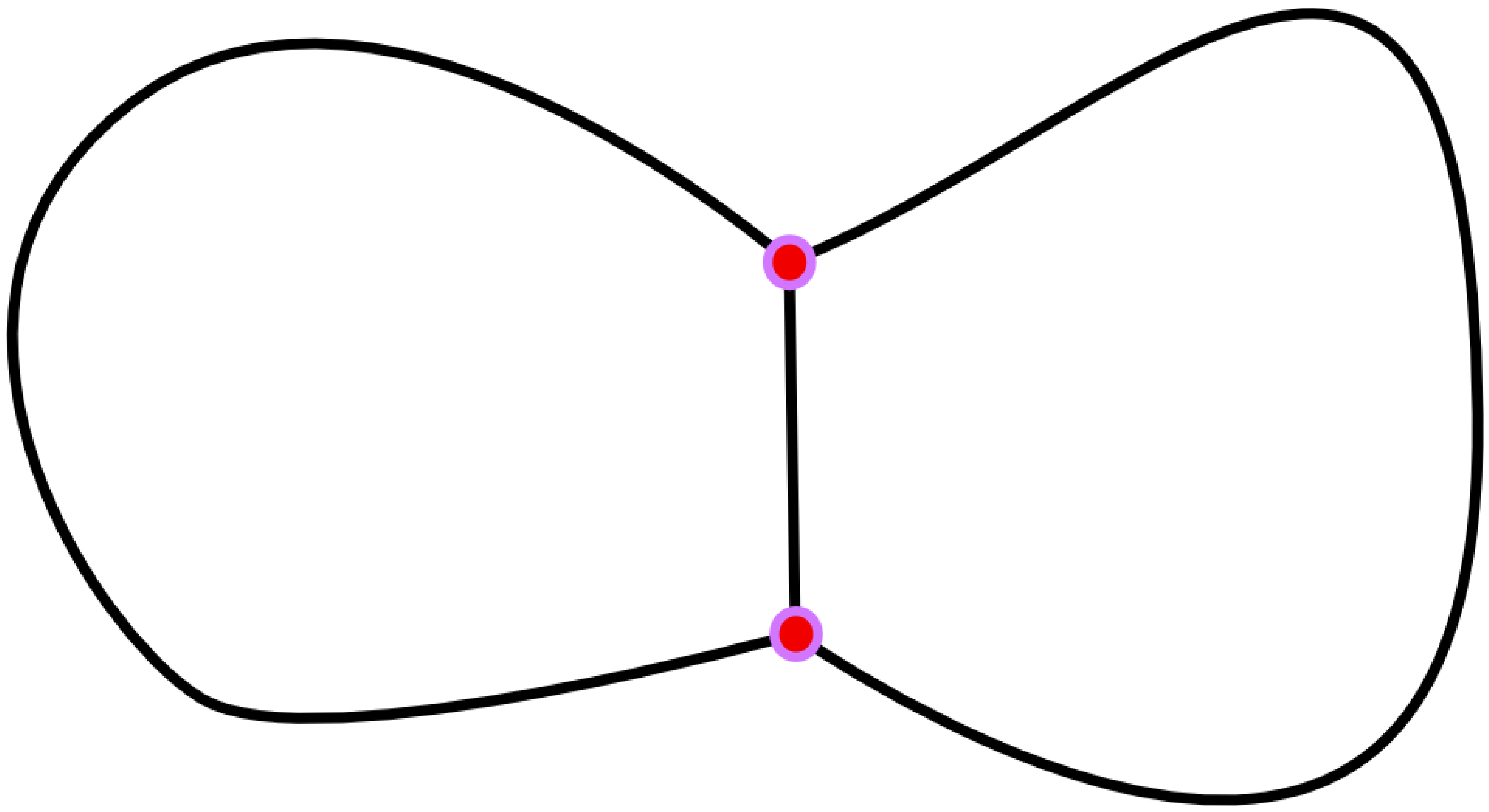


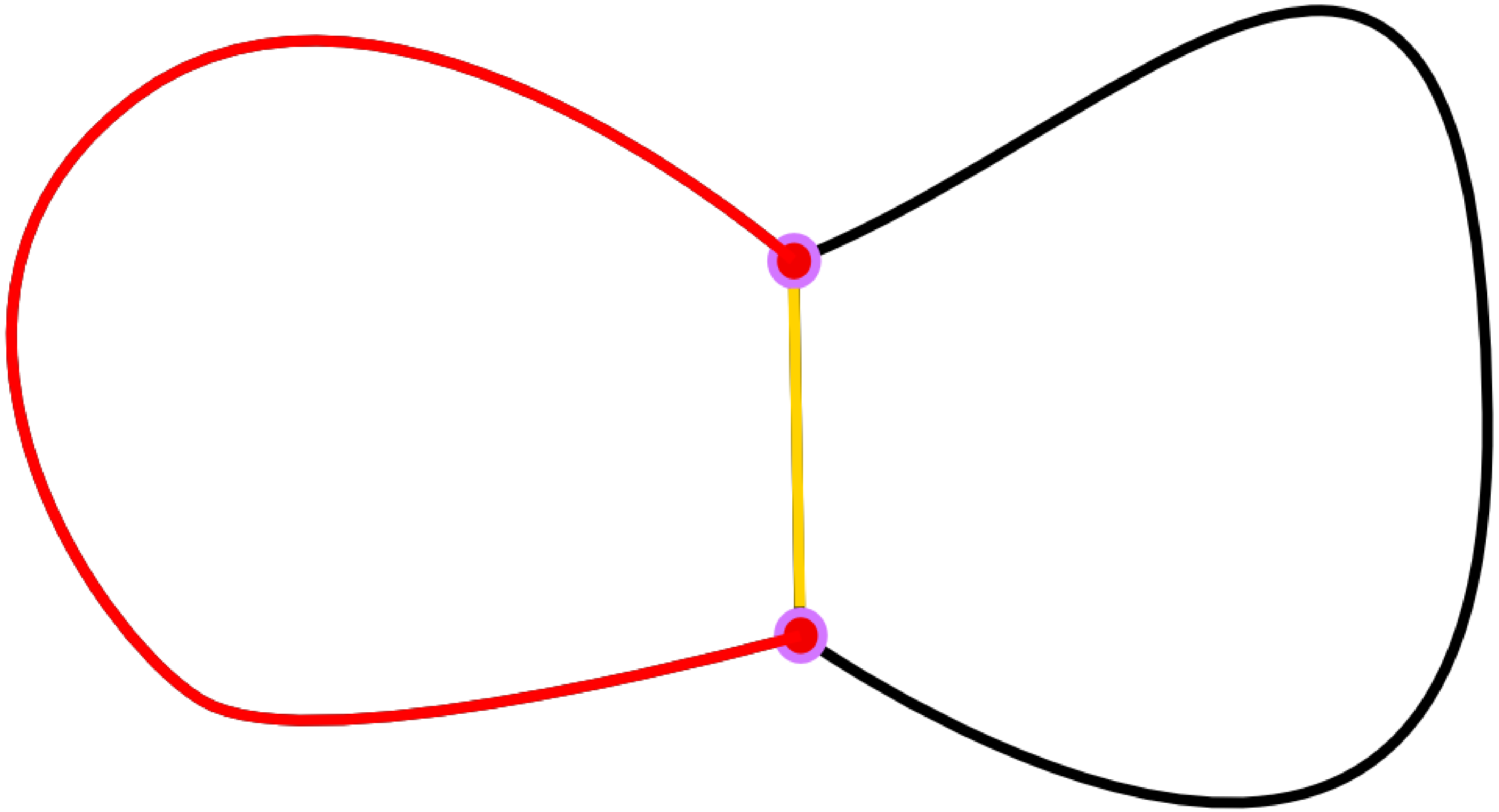
Cutting Graph

On a cutting graph G we locate the *Branching vertices*



Cutting Graph

A *segment* in a cutting graph G is a set of consecutive edges in G, which connect two branching vertices.



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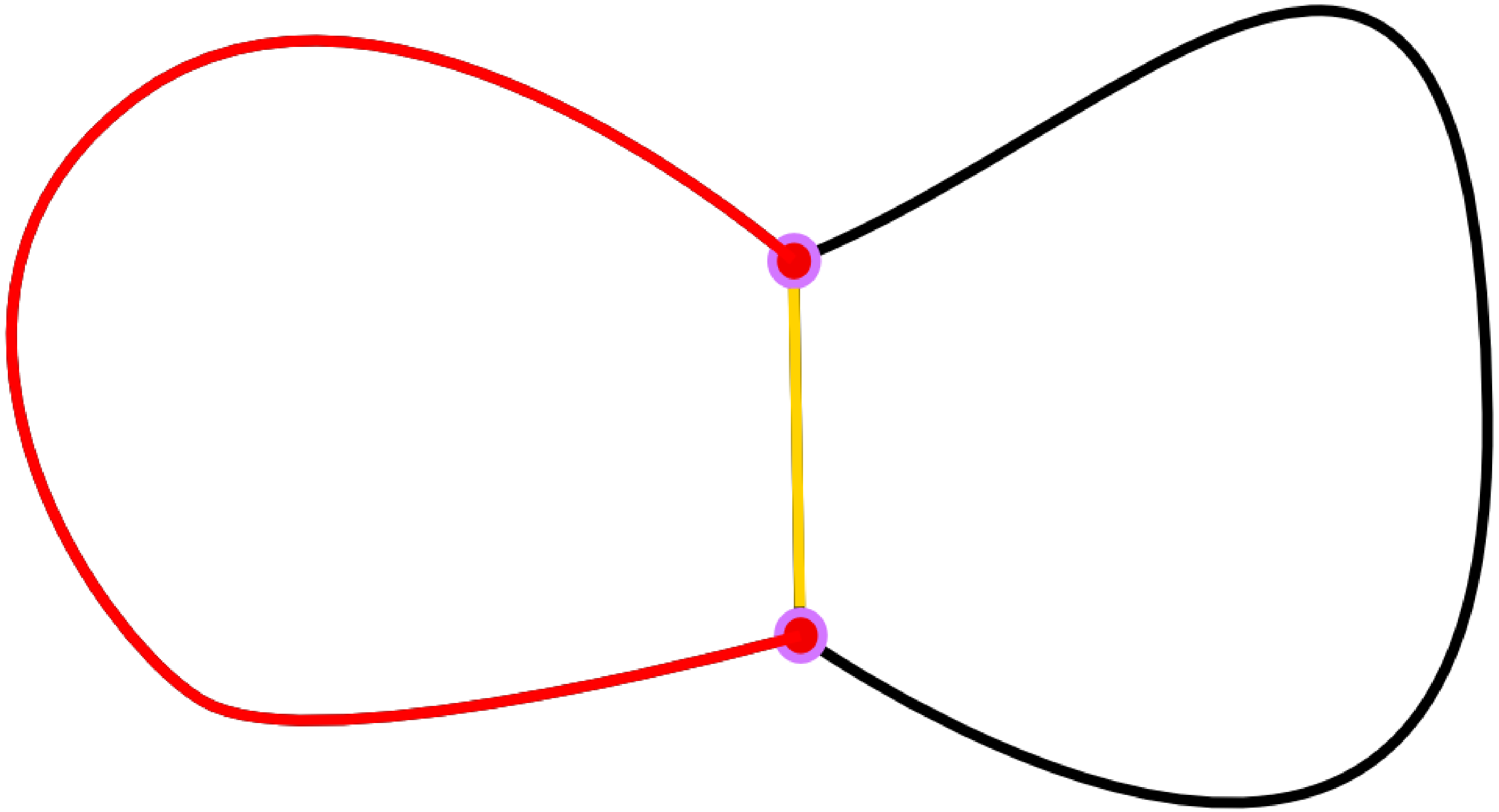
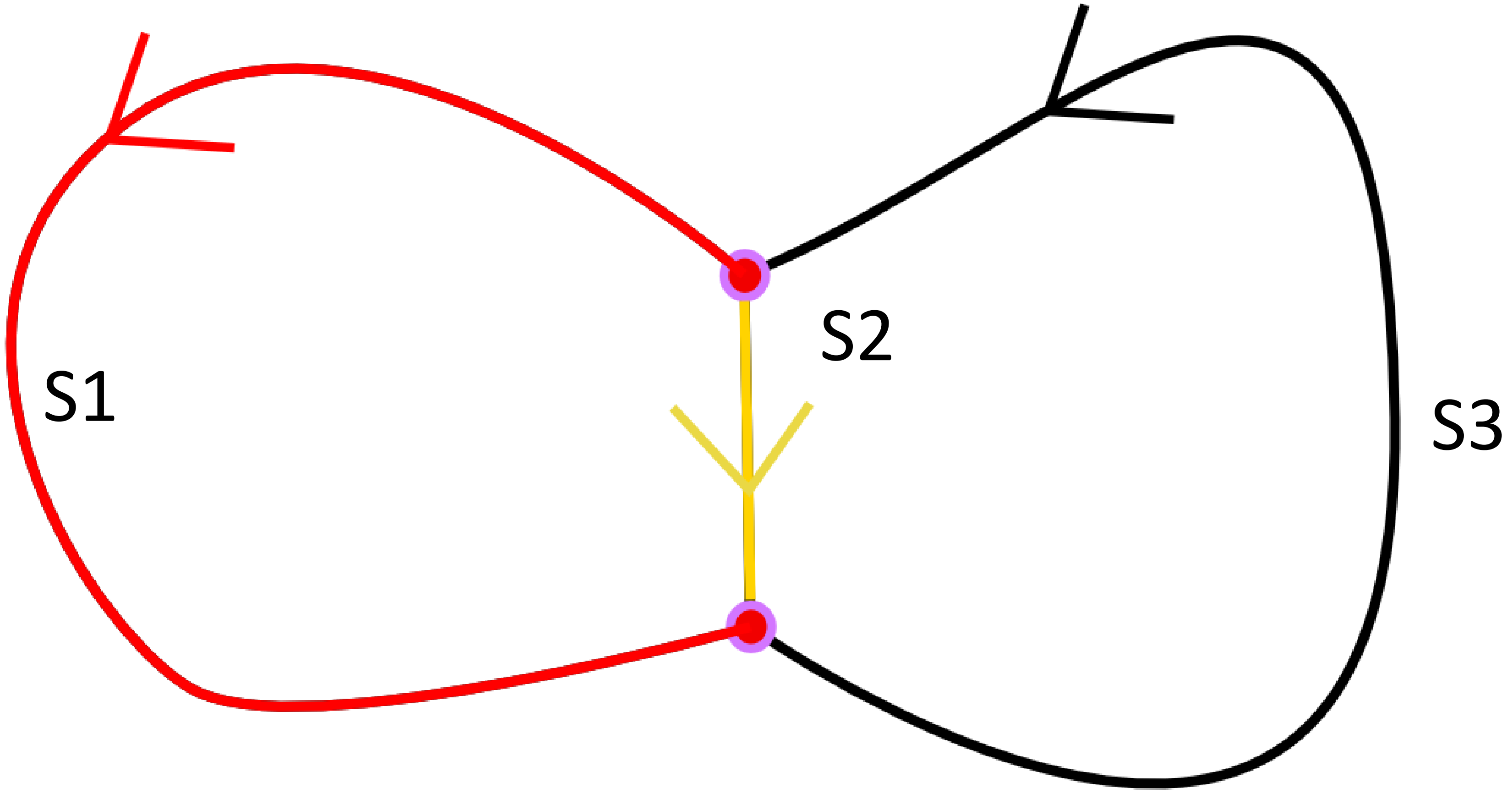


Namely the branching vertices separate the cut graph into a collection of segments.

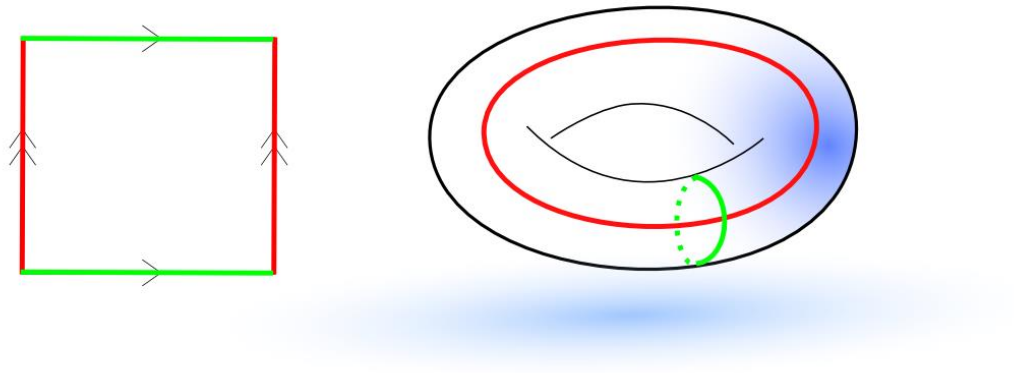
Cutting Graph

We give each segment an arbitrary orientation and denote the oriented segments by S={s_1,...,sn}

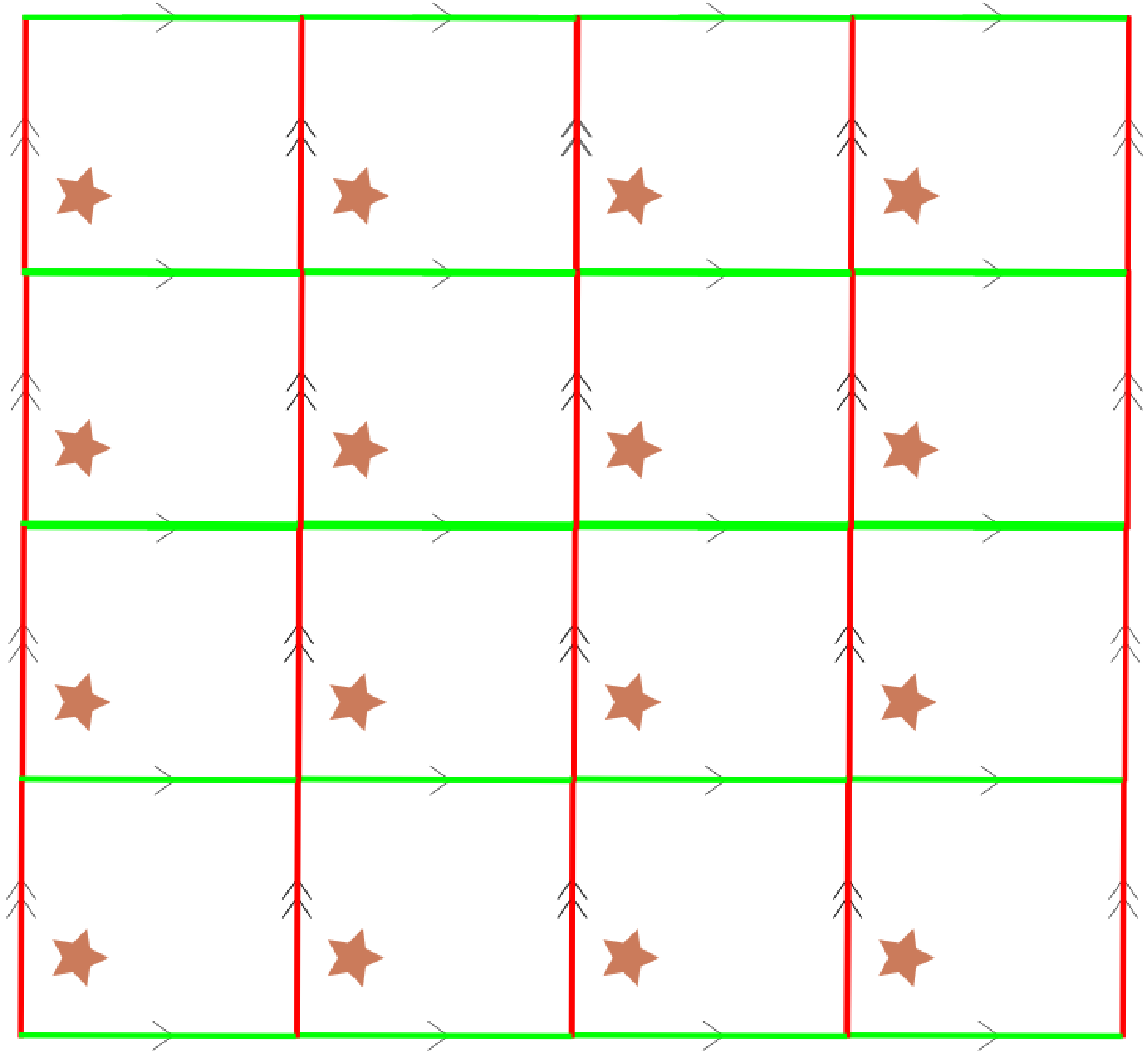


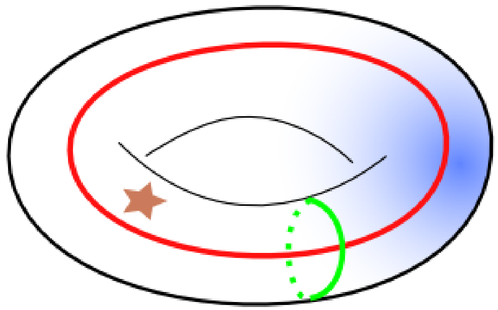
Computing the fundamental domain

A closed subset D of the universal cover $\tilde{M}$ of M is called a fundamental domain of $\tilde{M}$ if $\tilde{M}$ is the union of conjugates of D



The universal cover and the fundamental domain



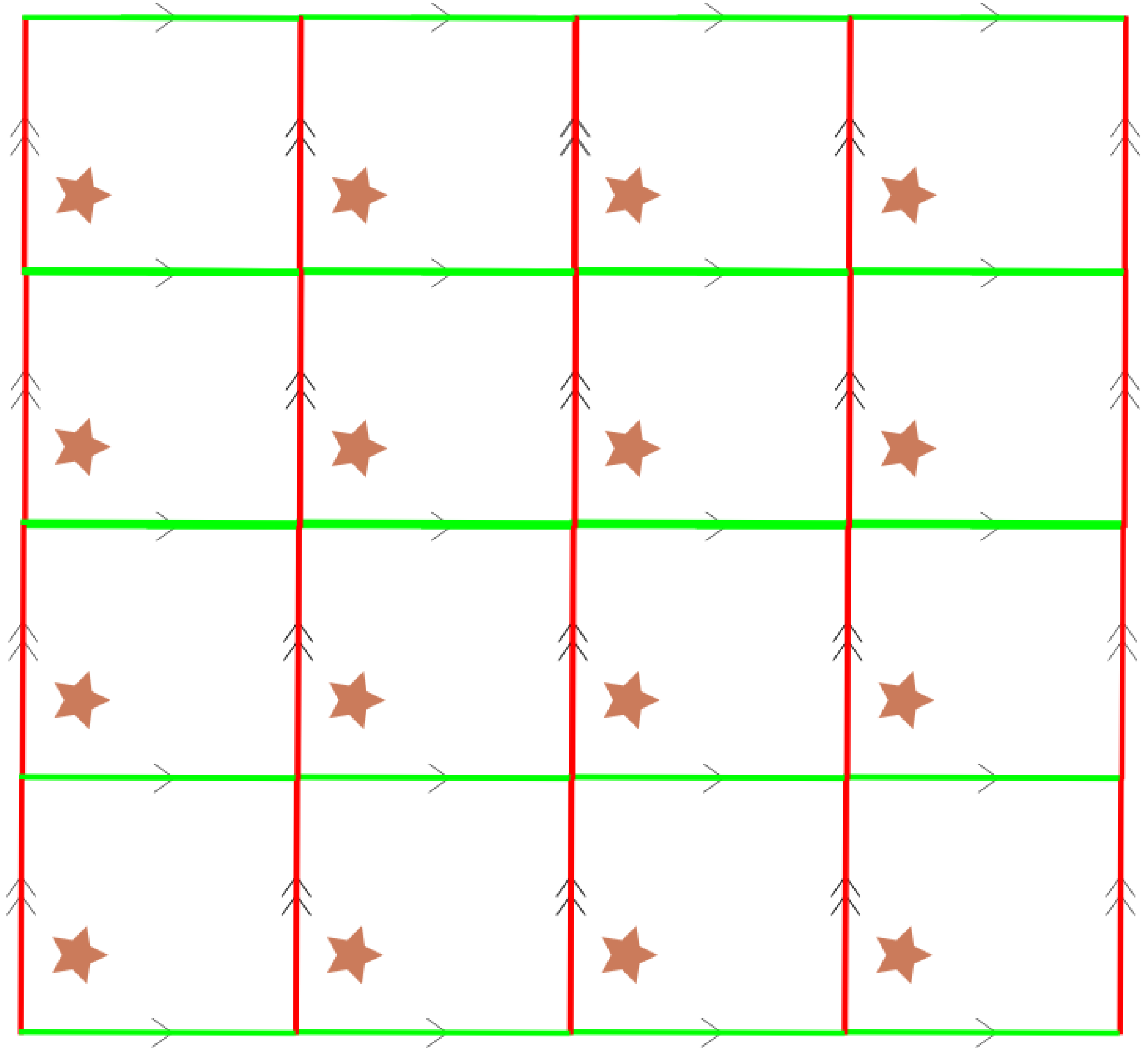


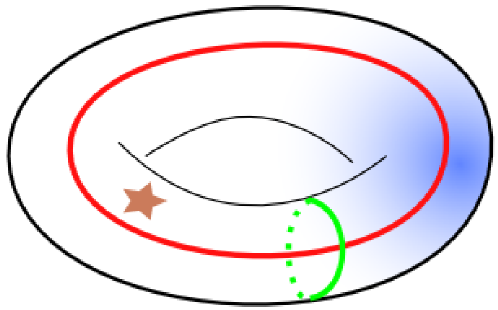
Computing the fundamental domain

Input : A mesh M
Output : A fundamental domain D of M

- Compute a cut graph G of M
- Slice M along G

The universal cover and the fundamental domain





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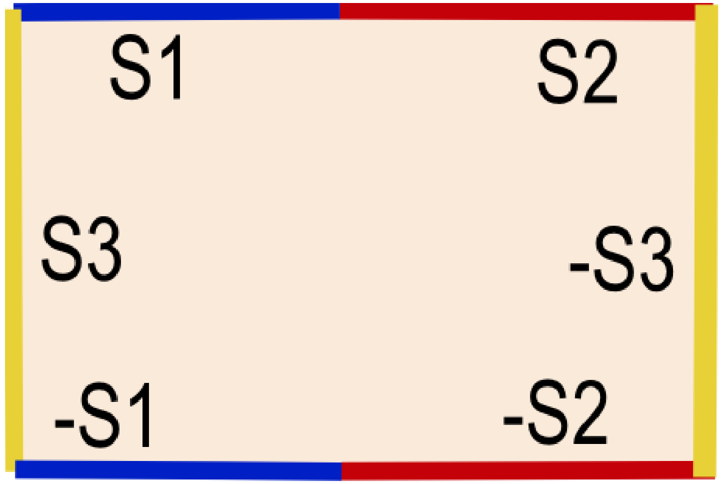

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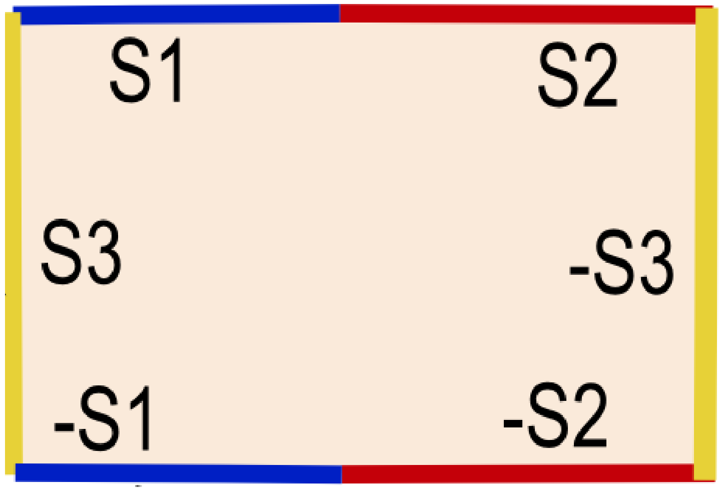

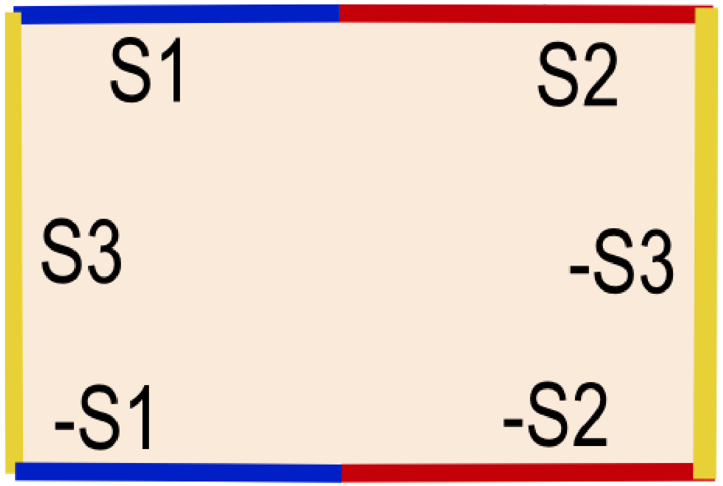

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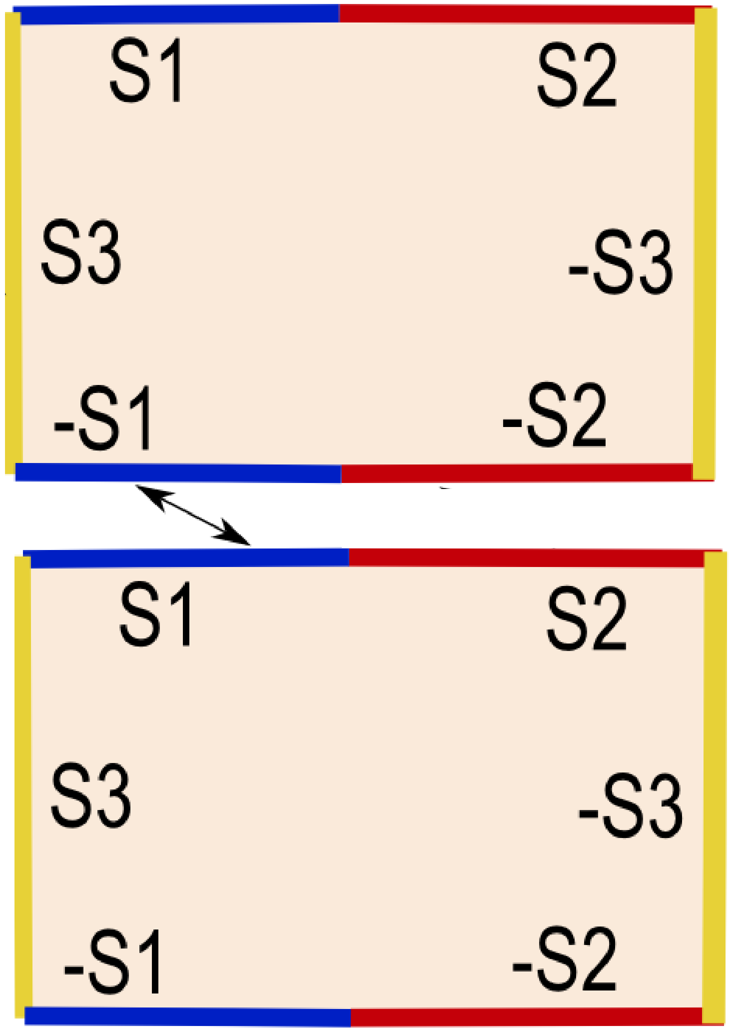

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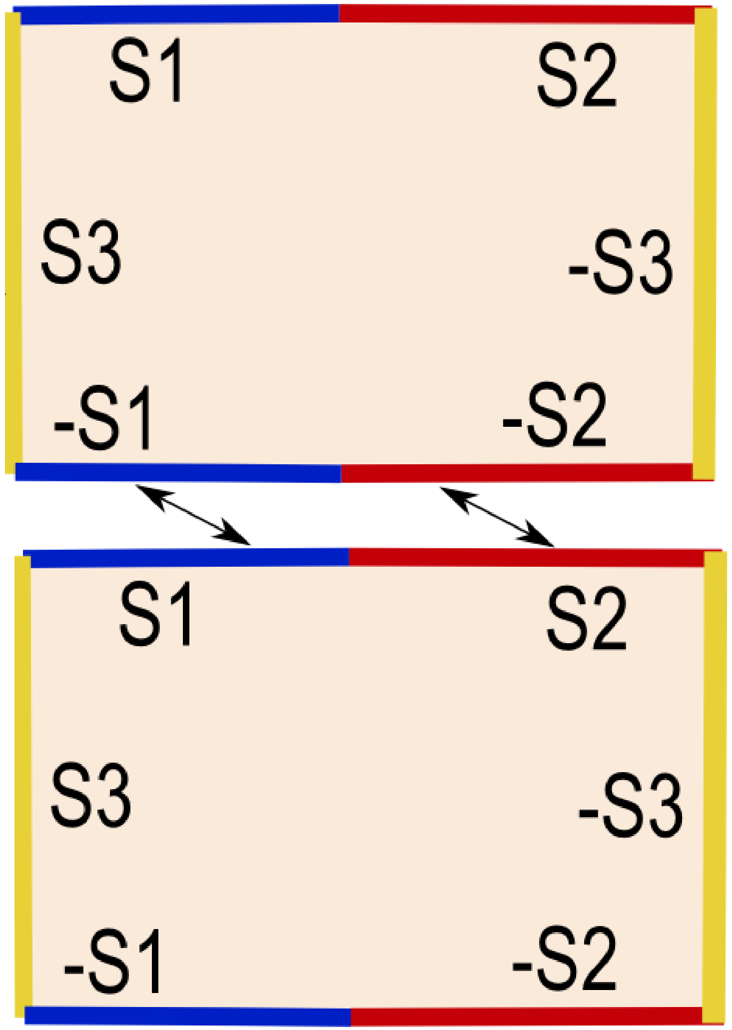

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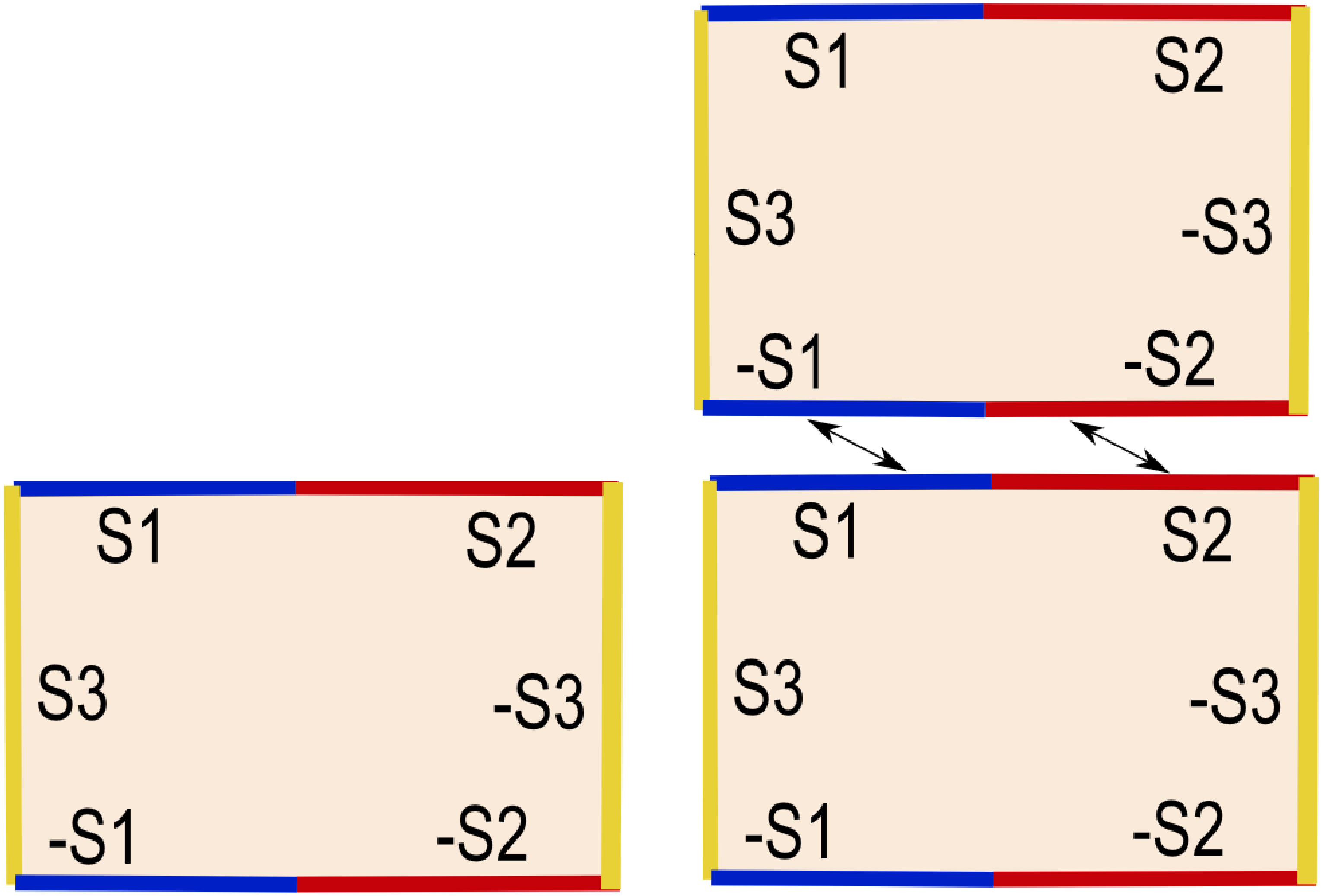

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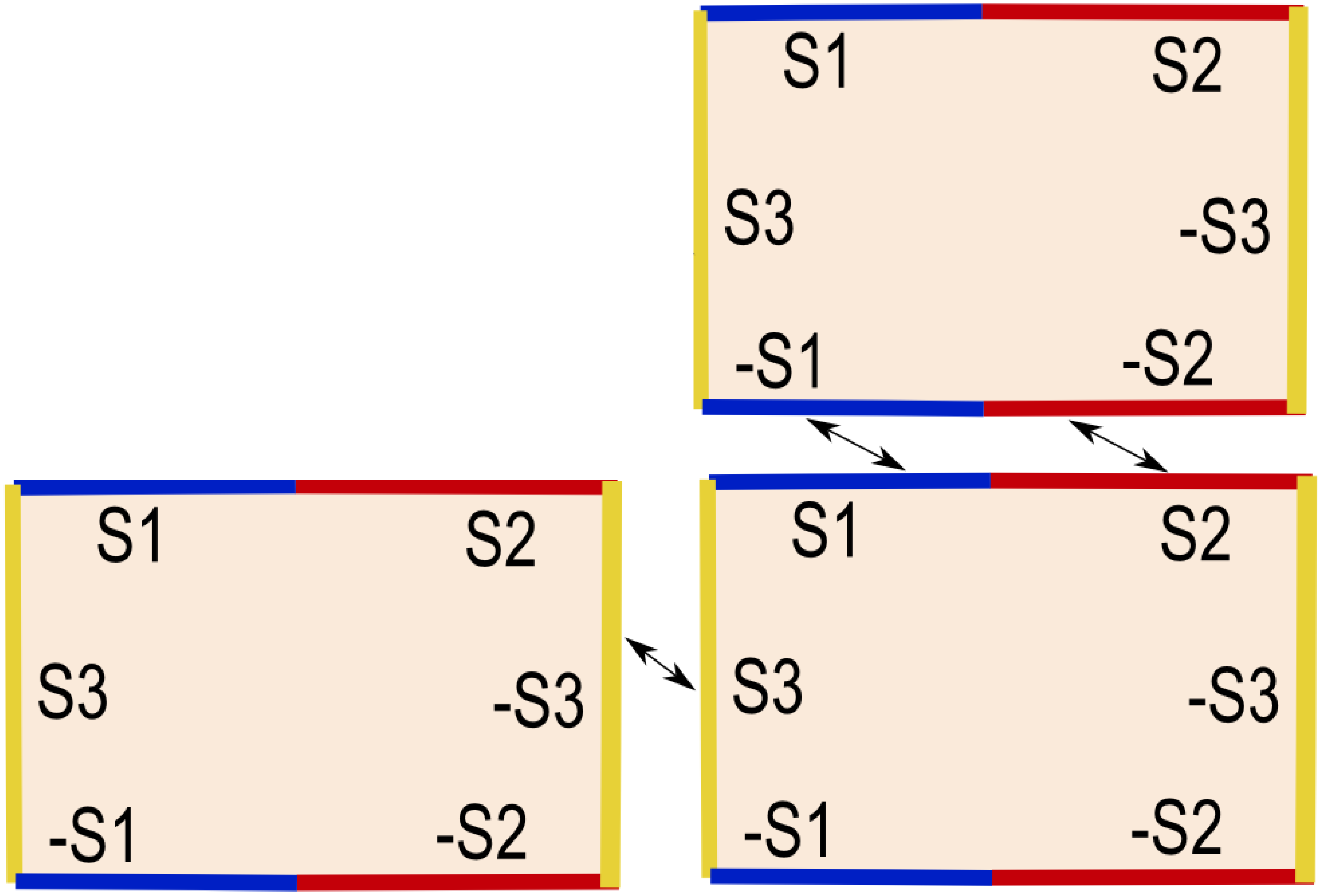

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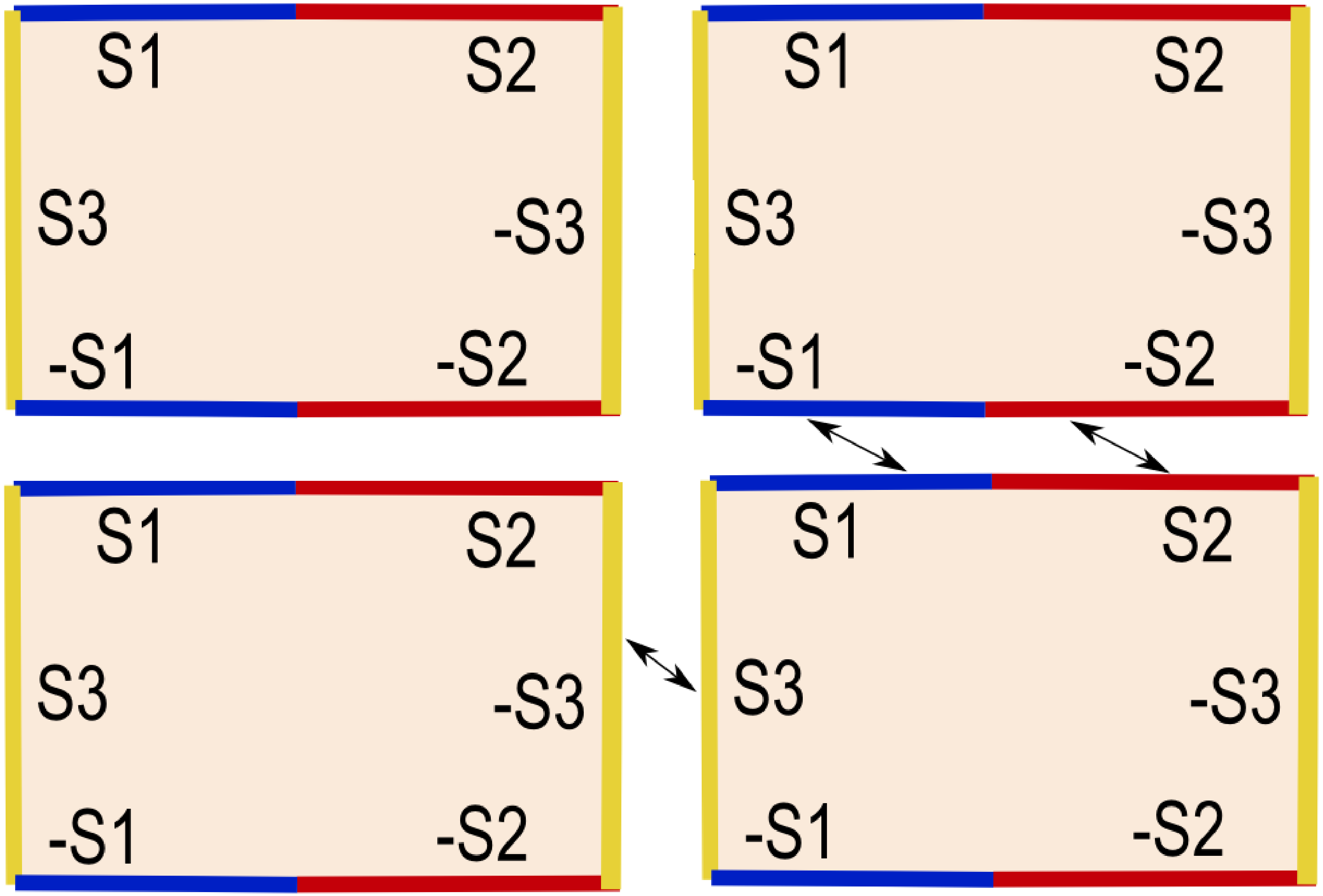

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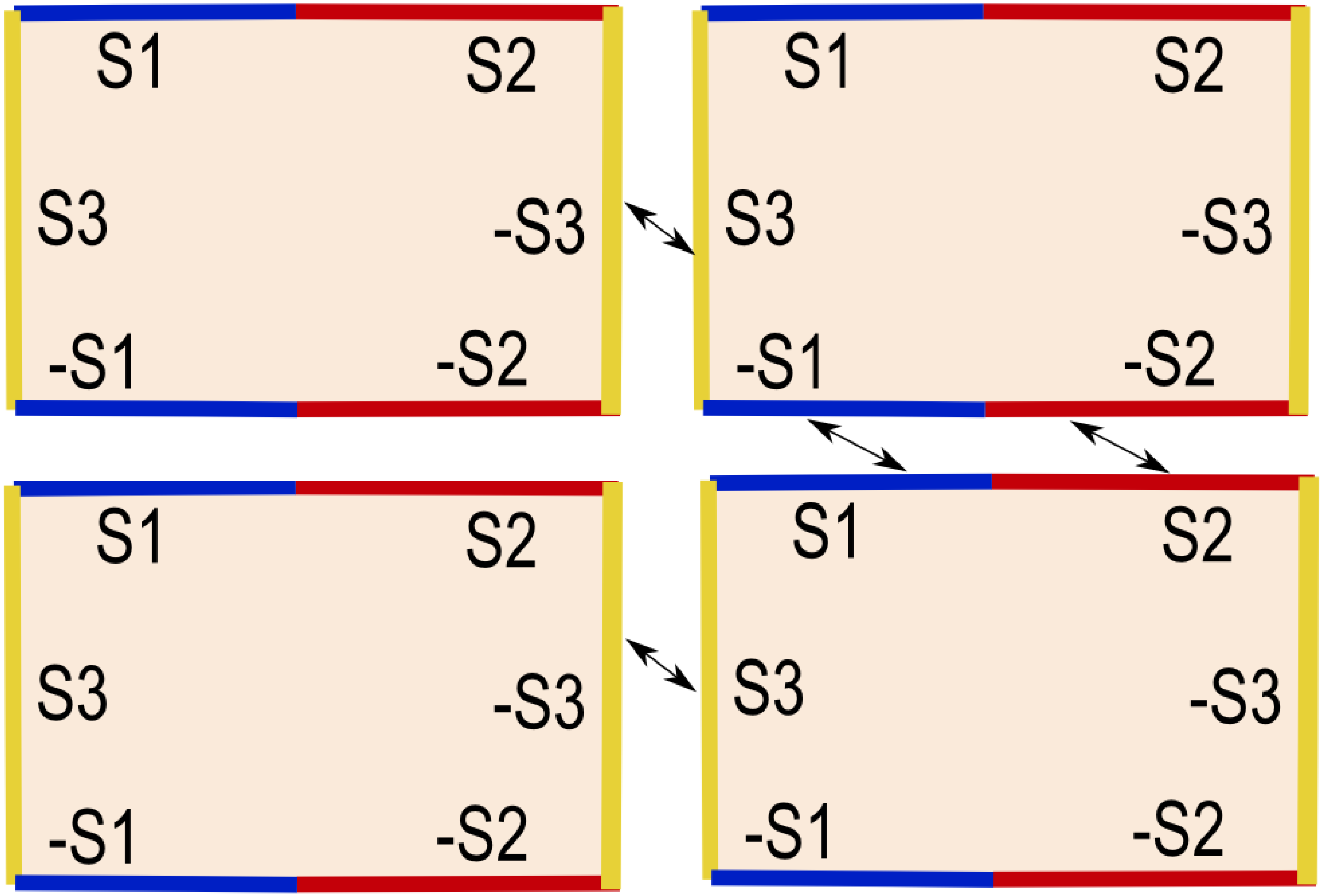

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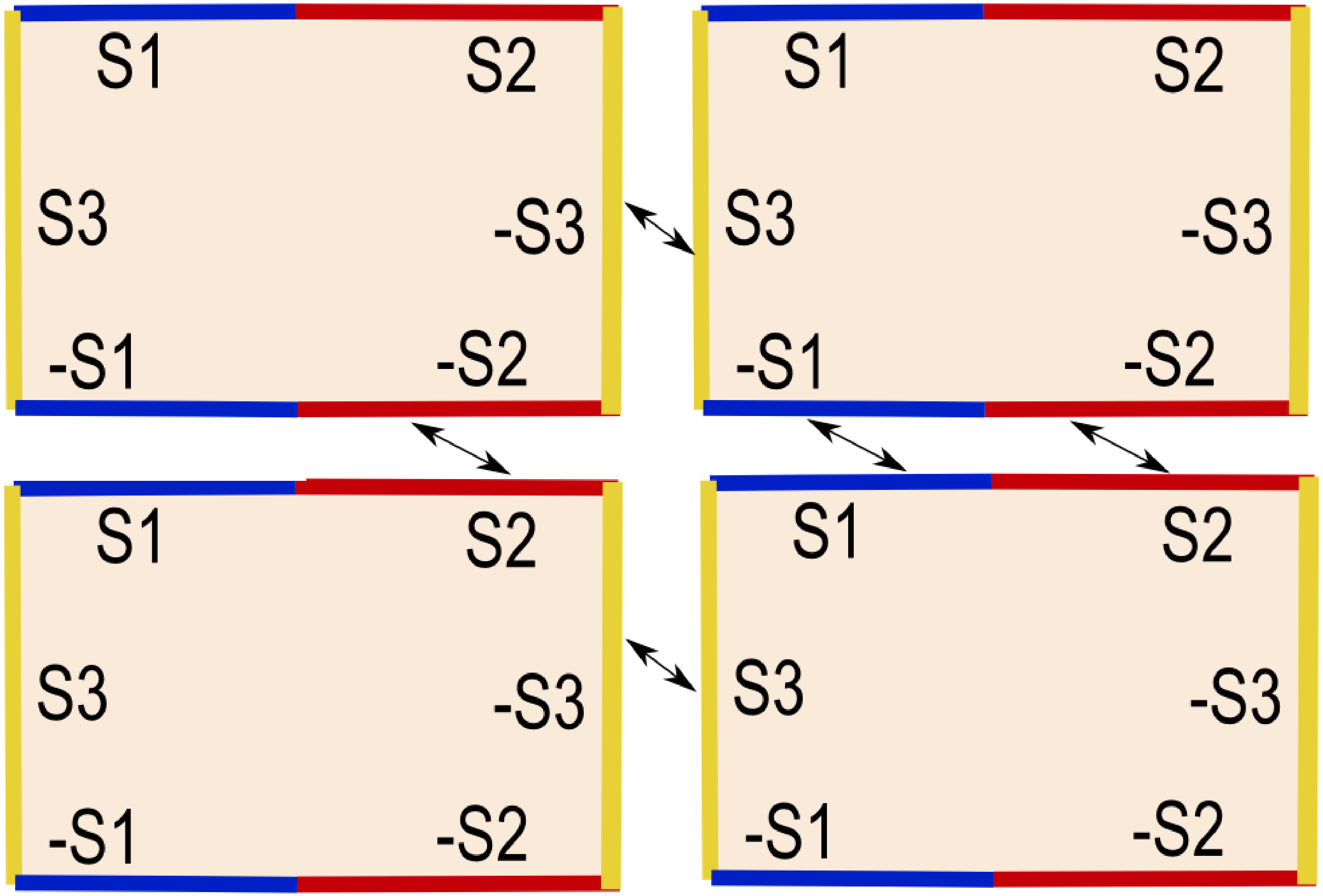

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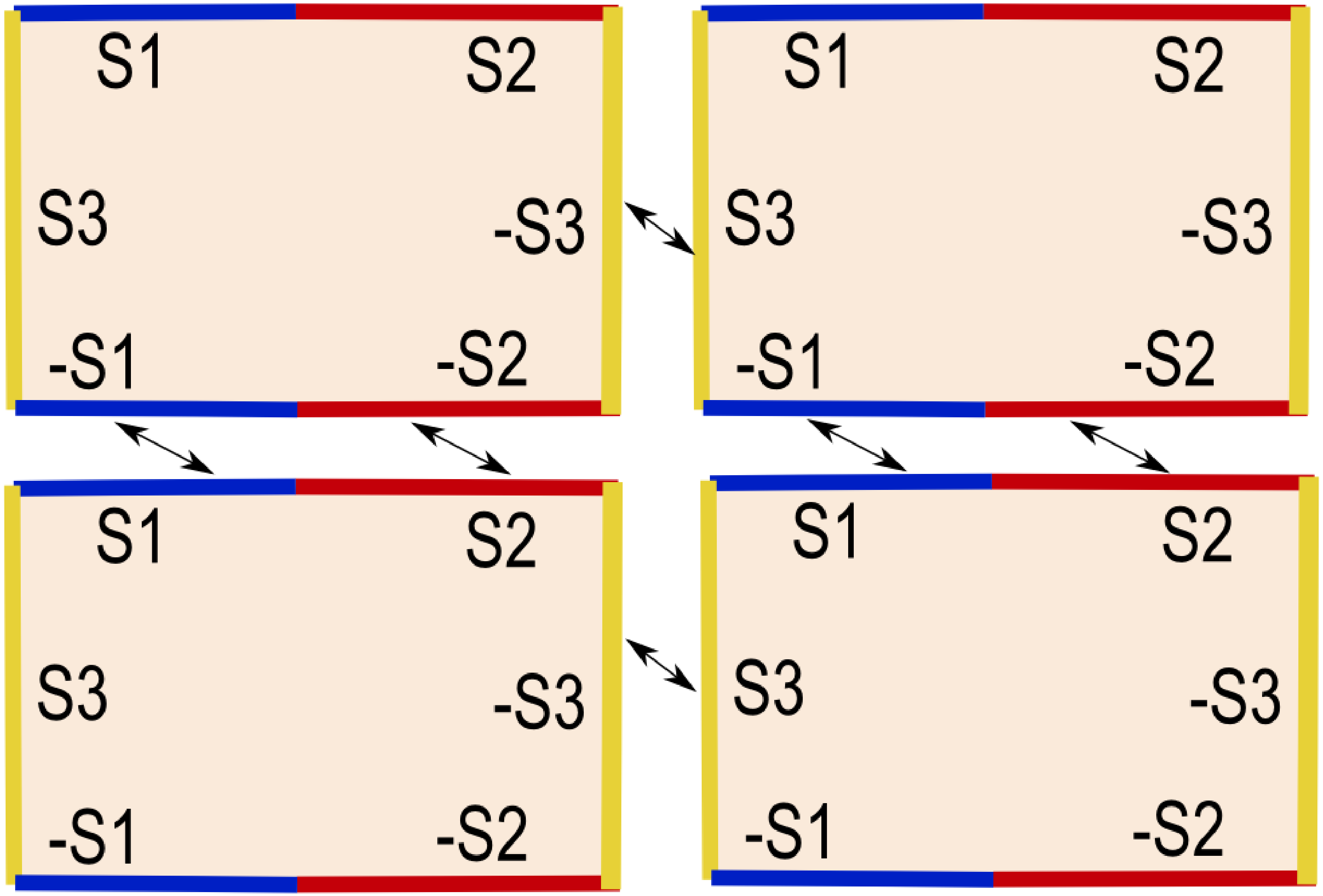

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Input : A mesh Σ.

Output: A finite portion of the universal cover Σ^{-}.

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We call a vertex on G with valence greater than 2 a *Branching vertex*. The *Branching vertices* divide G to segments, assign an orientation to each segment, labeled as {s1, s2, · · · , sn}.

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5 Update $\partial\bar{\Sigma}$, if $\pm si$ are adjacent in $\partial\bar{\Sigma}$, glue the boundary of $\bar{\Sigma}$ along si . Repeat this step until no adjacent $\pm si$ in the boundary.

Constructing the universal cover

Input : A mesh Σ.
Output: A finite portion of the universal cover Σ¯.
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We call a vertex on G with valence greater than 2 a *Branching vertex*. The *Branching vertices* divide G to segments, assign an orientation to each segment, labeled as {s1, s2, · · · , sn}.

2 Slice Σ along G to get a fundamental domain D, the boundary is composed of ±sk 's.

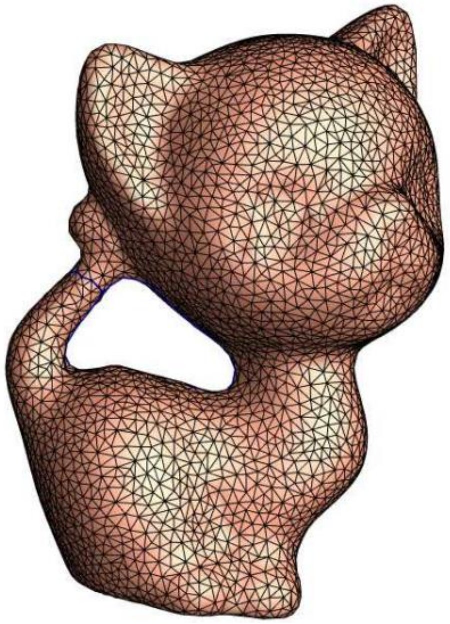
3 Initialize Σ¯ ← D, book keep ∂Σ¯ using ±sk 's.

4 Glue a copy of D to current Σ¯ along only one segment sk ∈ ∂Σ¯, −sk ∈ ∂D, Σ¯ ← Σ¯ $\cup_{sk}$ D.

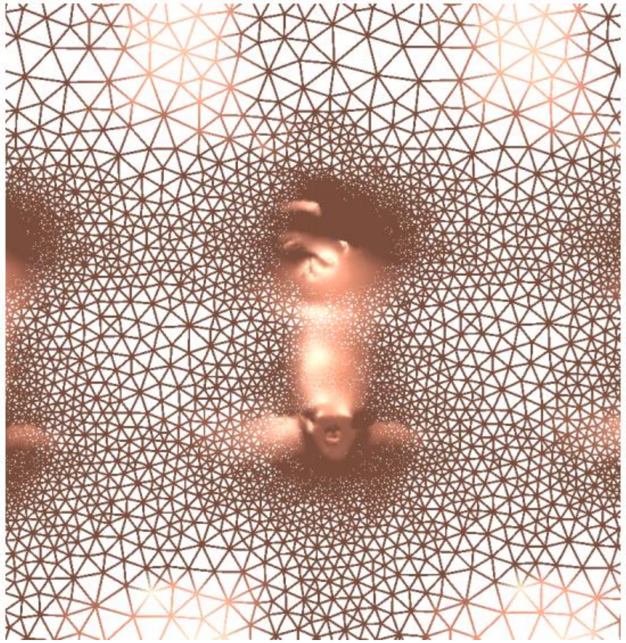
5 Update ∂Σ¯, if ±si are adjacent in ∂Σ¯, glue the boundary of Σ¯ along si . Repeat this step until no adjacent ±si in the boundary.

6 Repeat gluing the copies of D until Σ¯ is large enough.

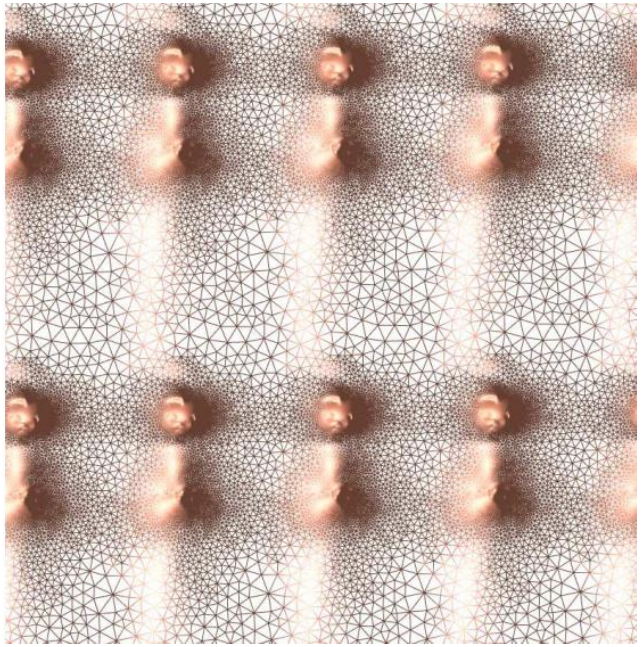
Constructing the universal cover



Input mesh



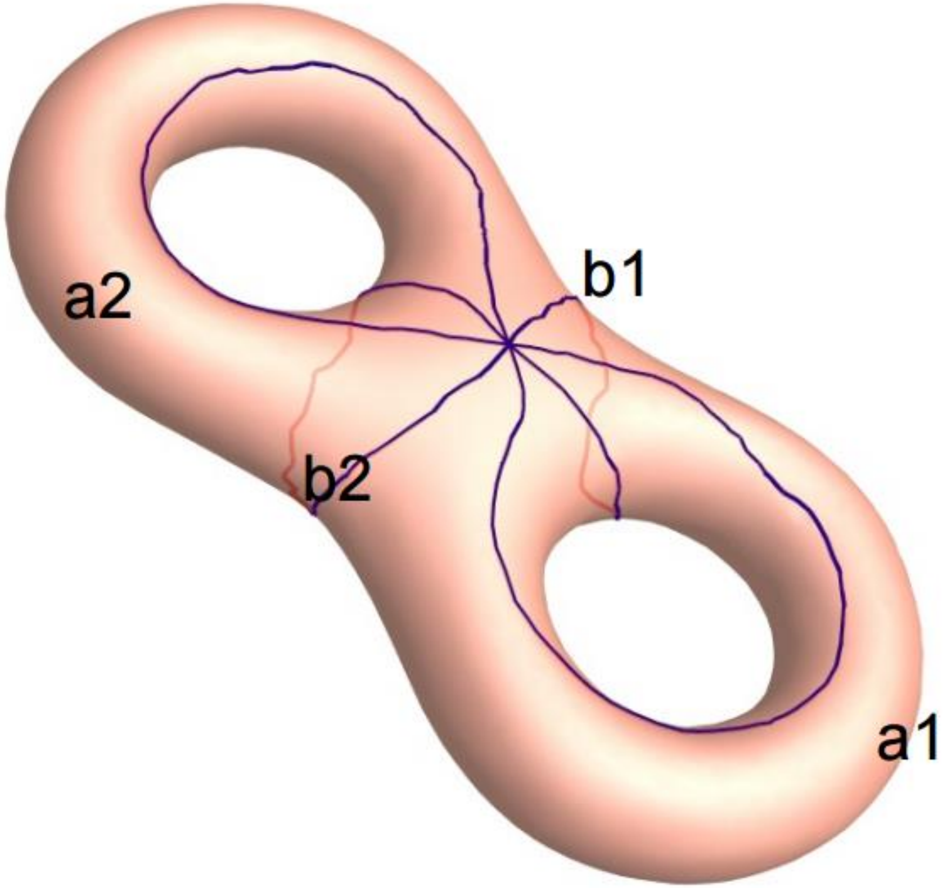
Fundamental domain

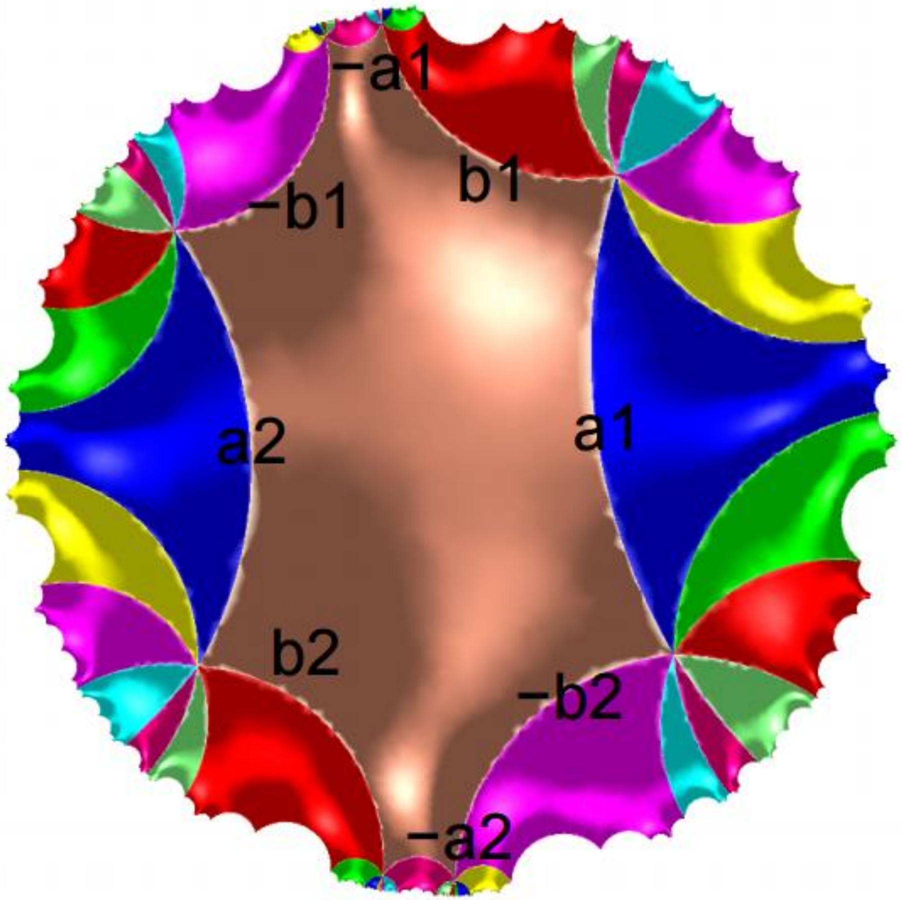


Finite portion of the universal cover

Image :David Gu

Constructing the universal cover





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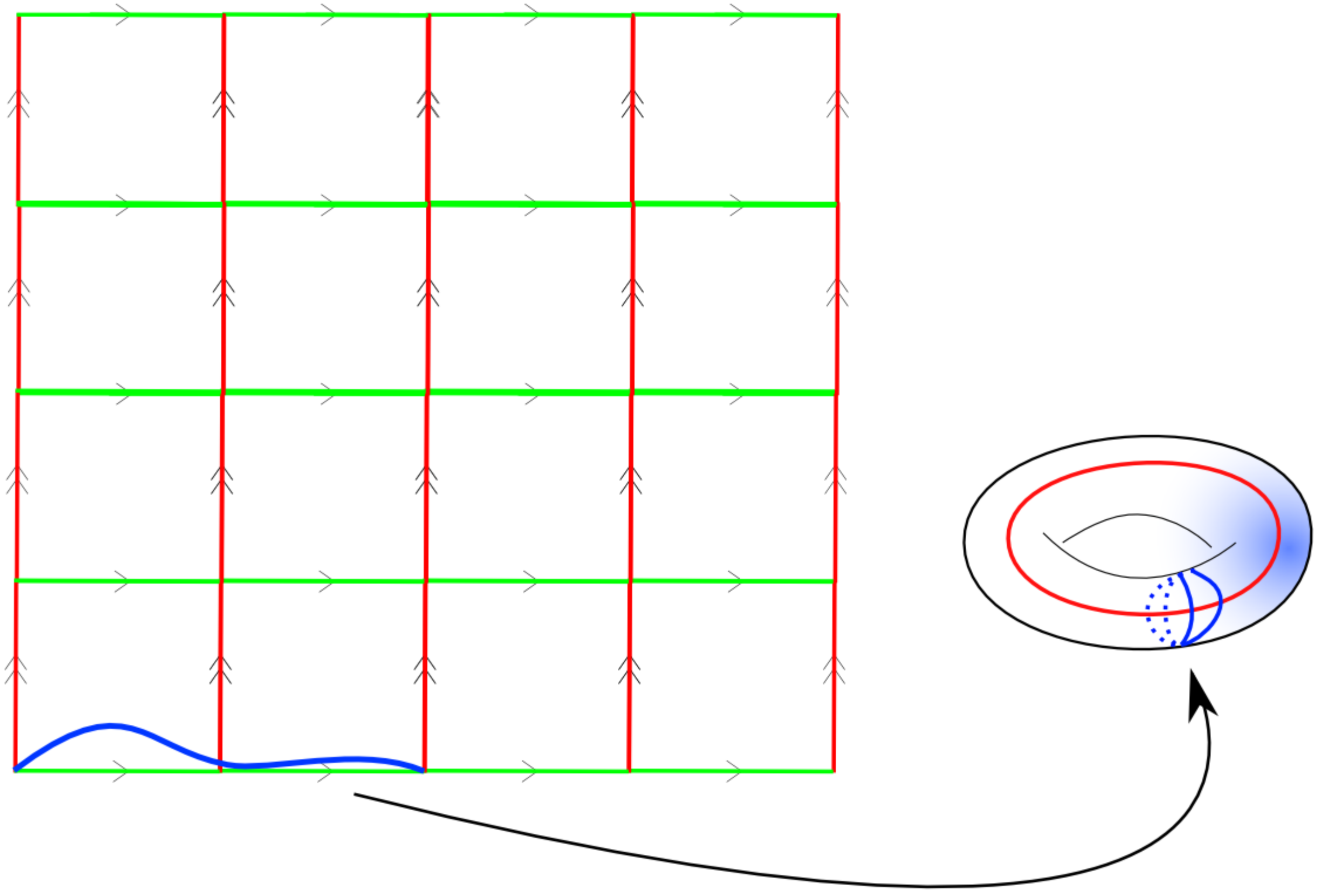

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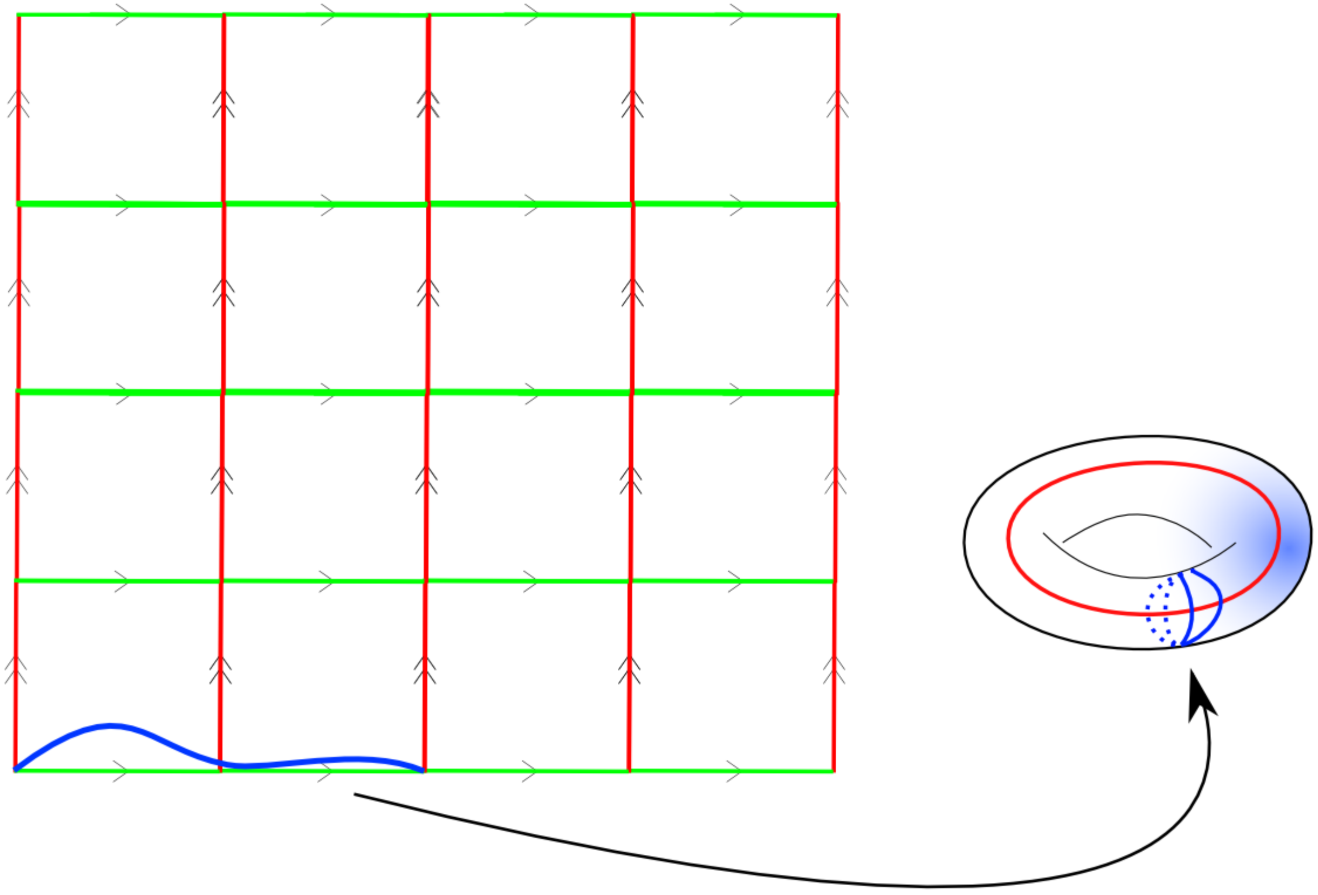

Key idea : many topological problems can be solved on the universal cover easier than on the original surface.

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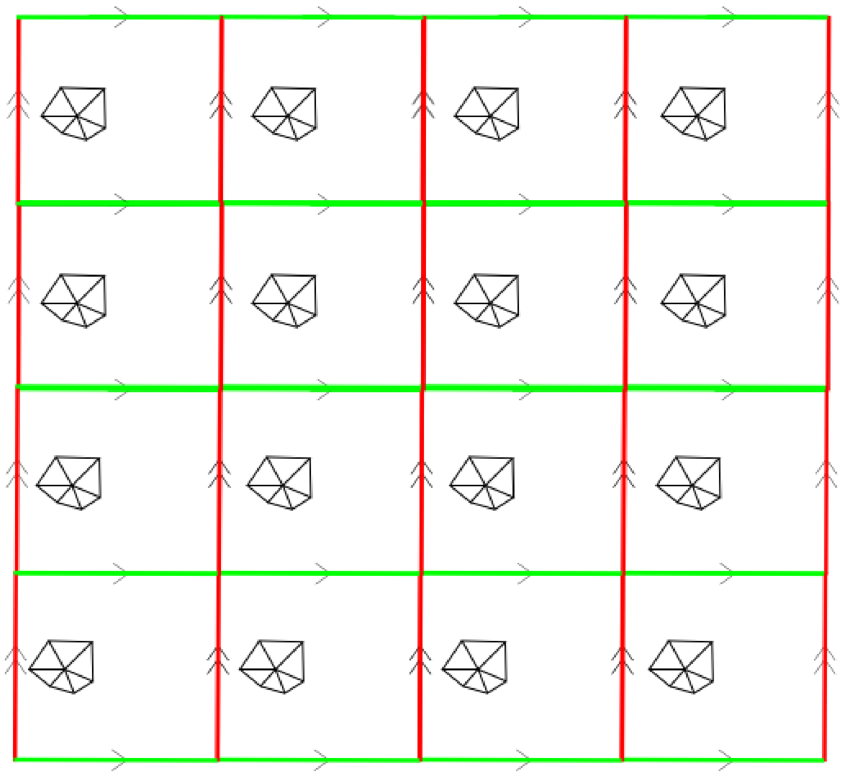

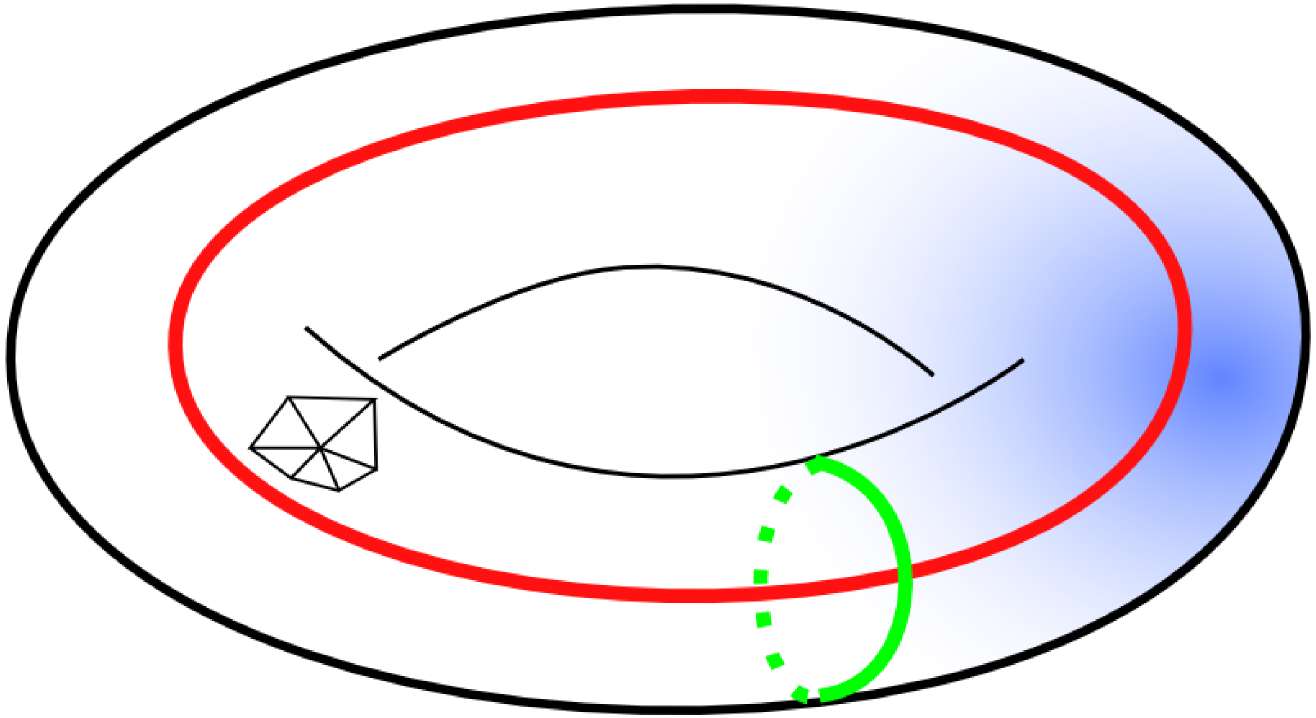

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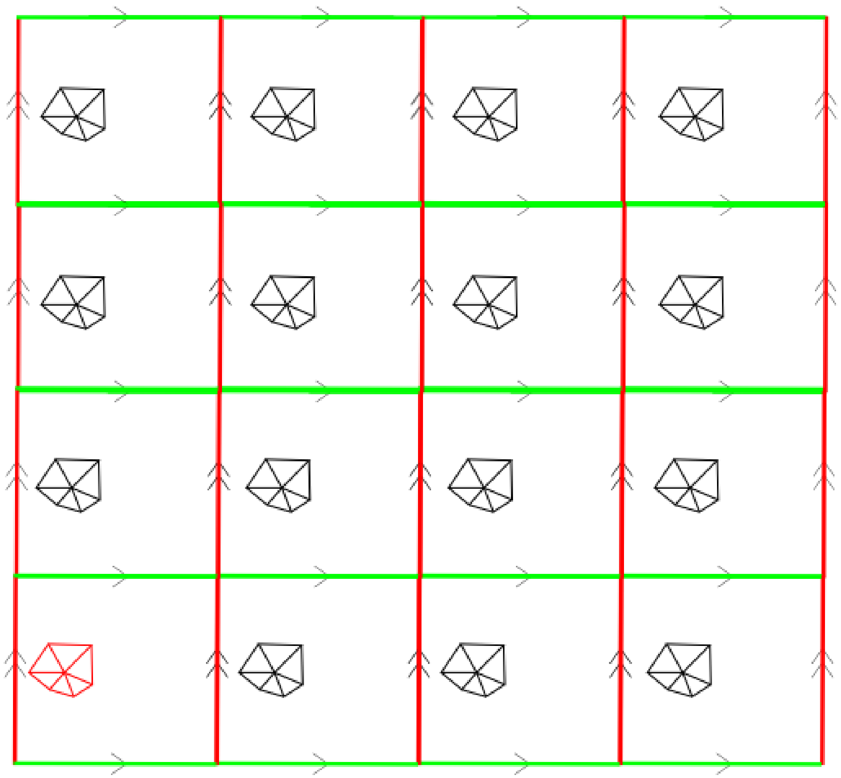

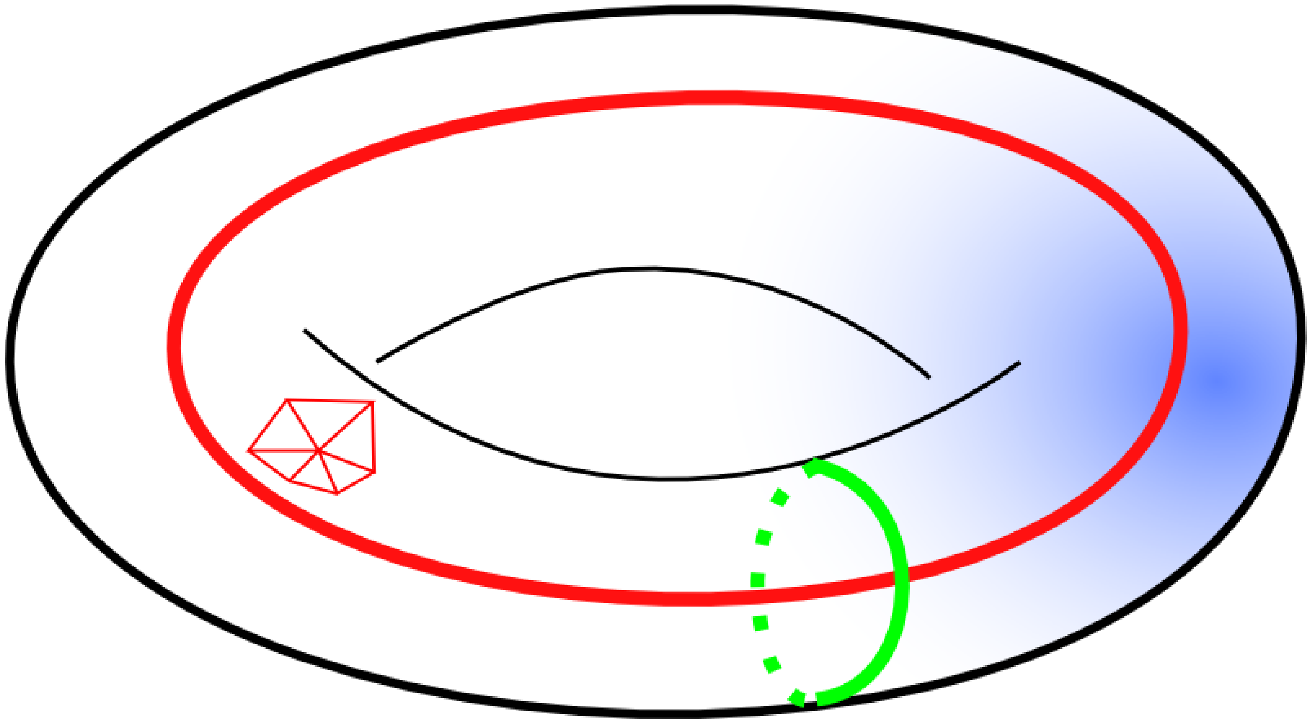

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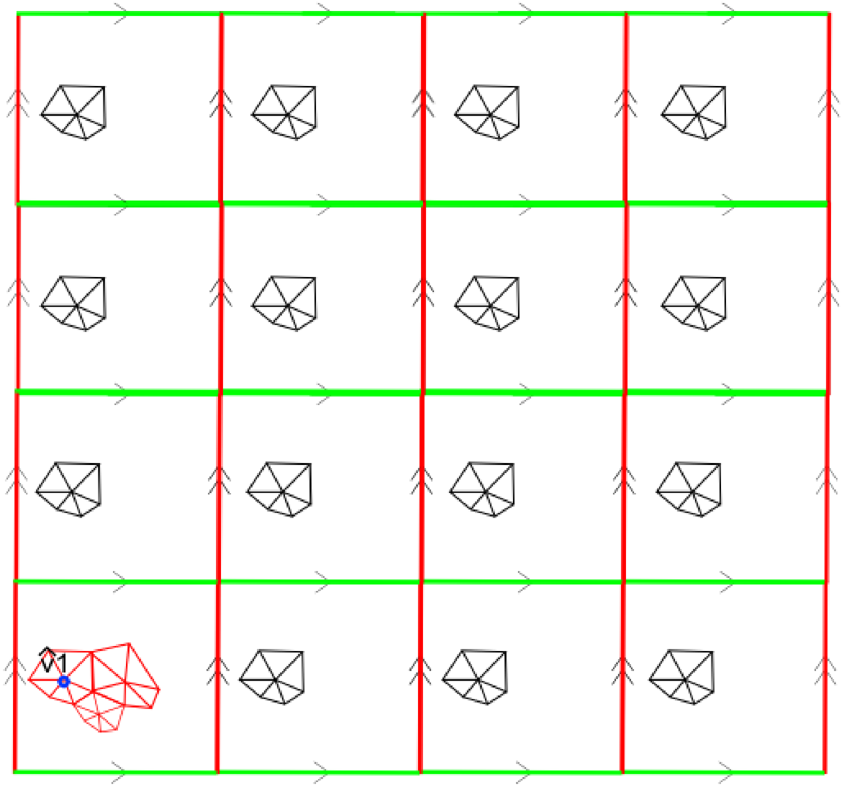

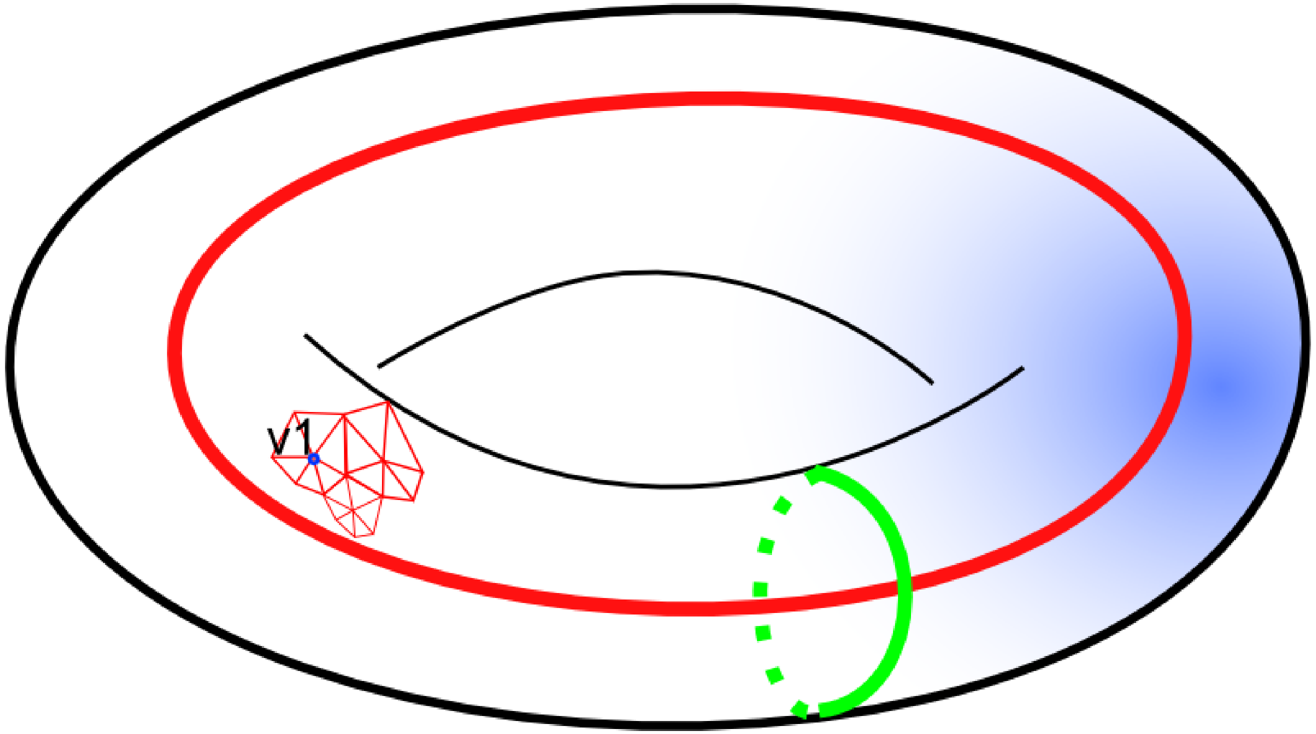

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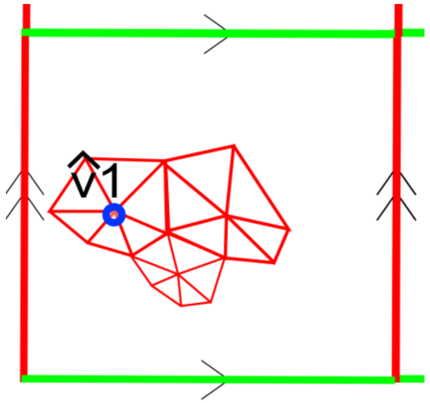

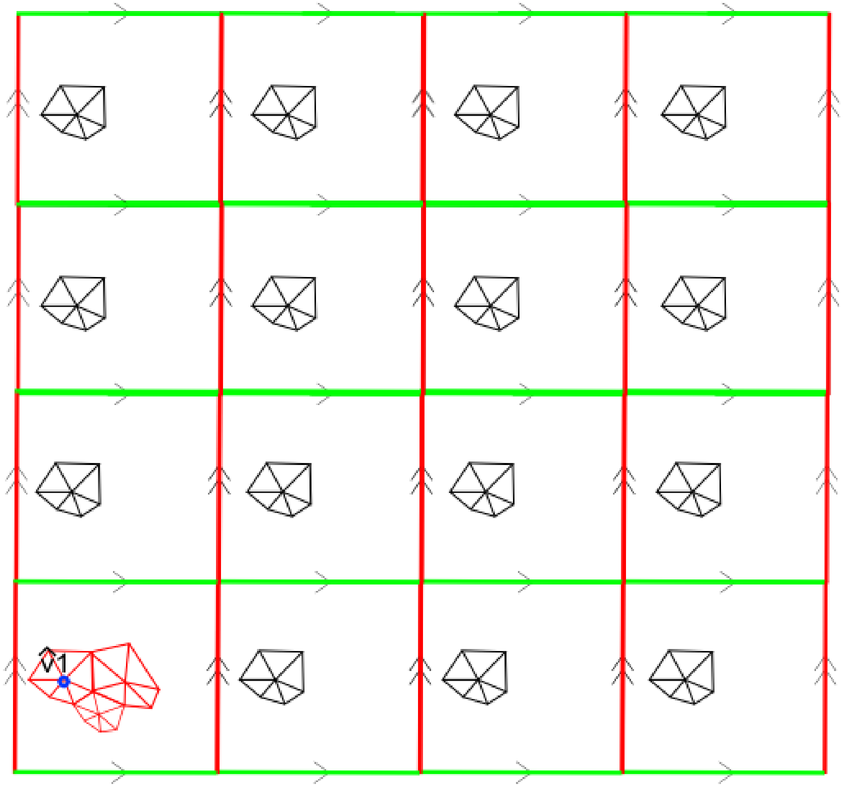

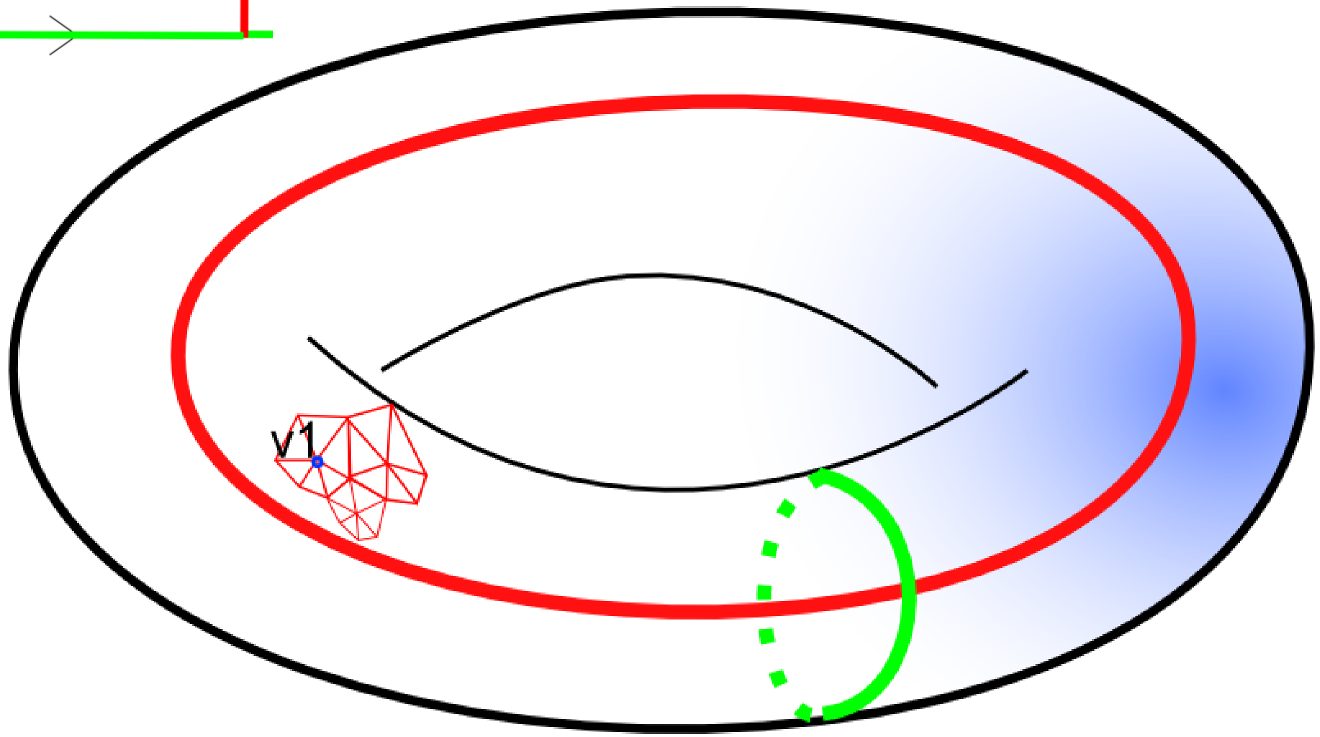

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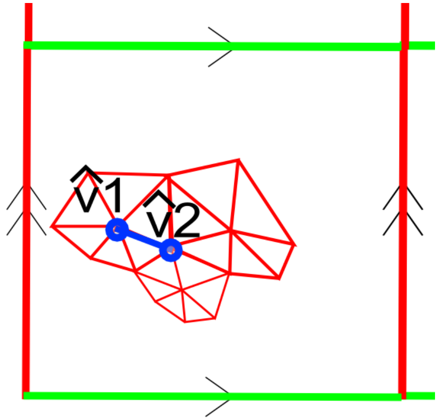

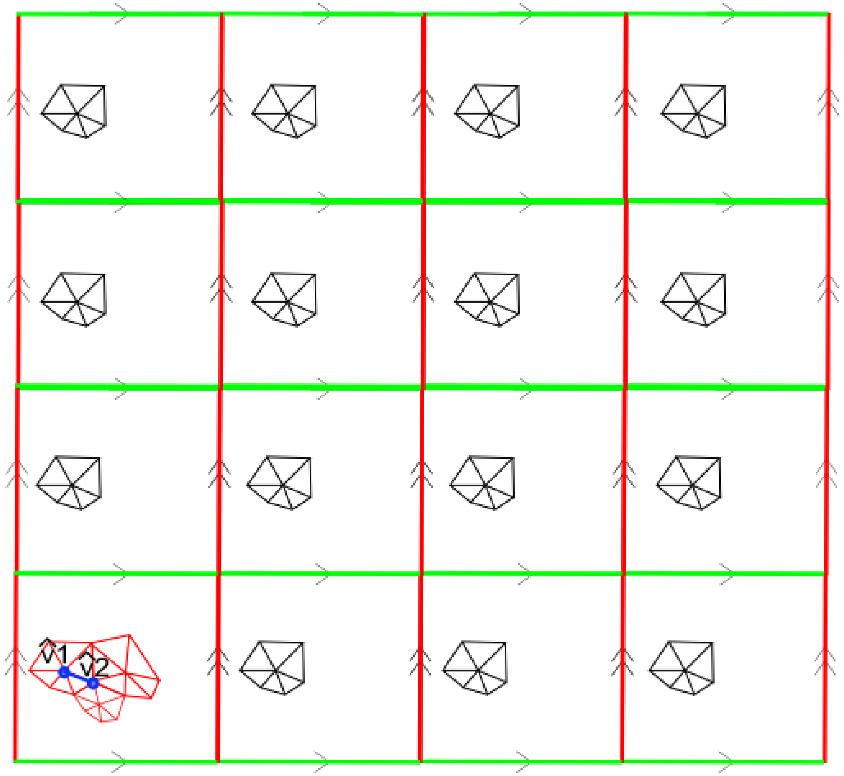

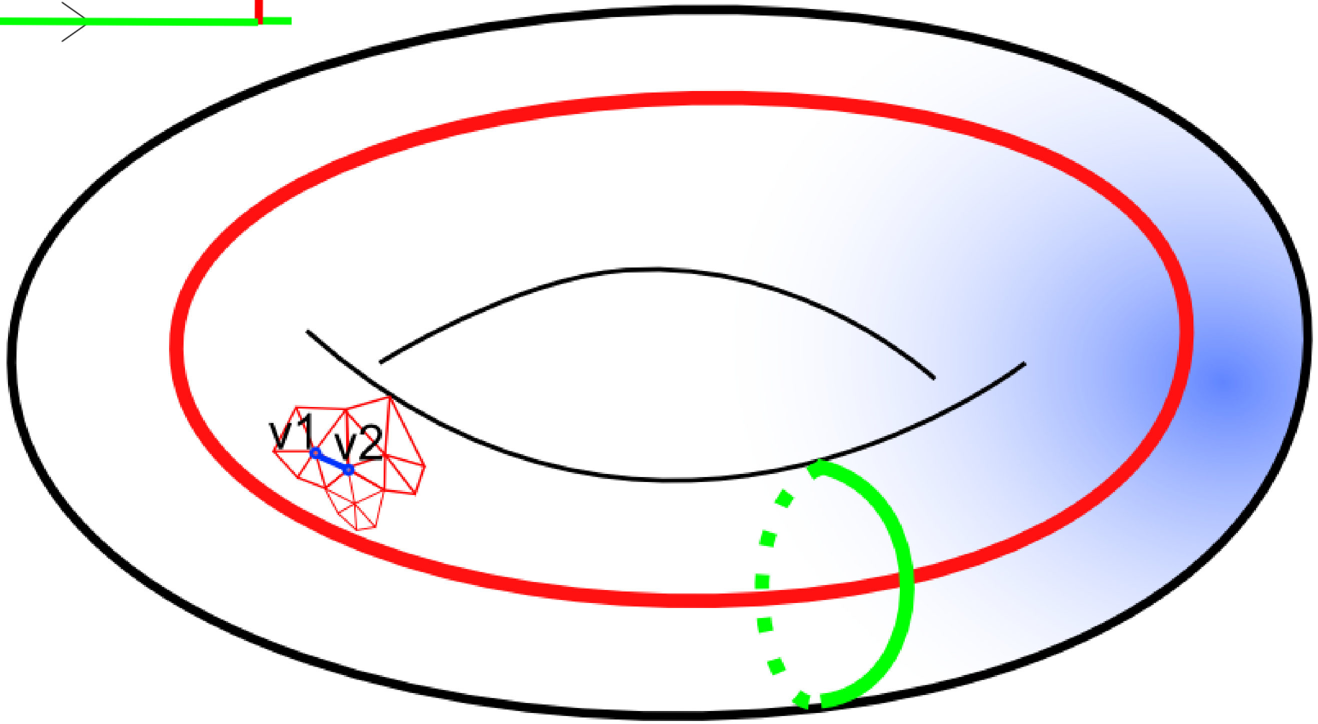

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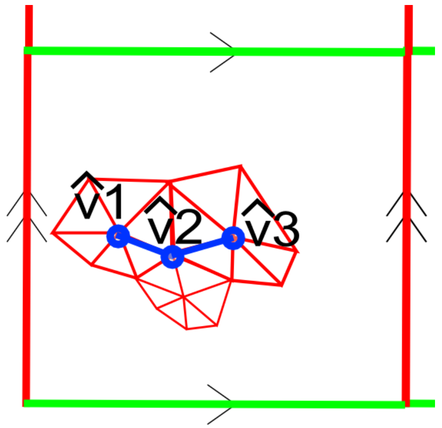

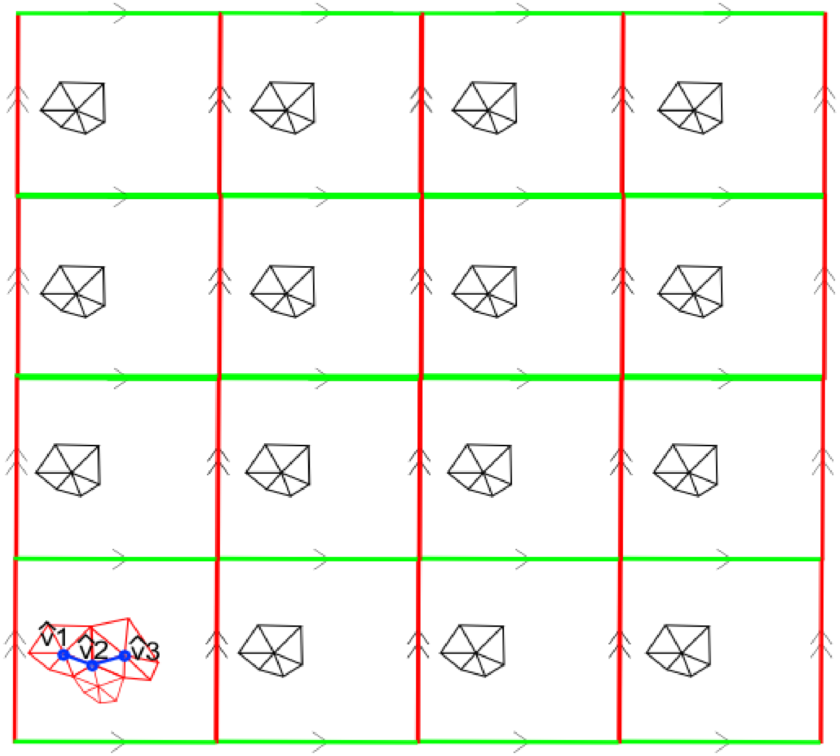

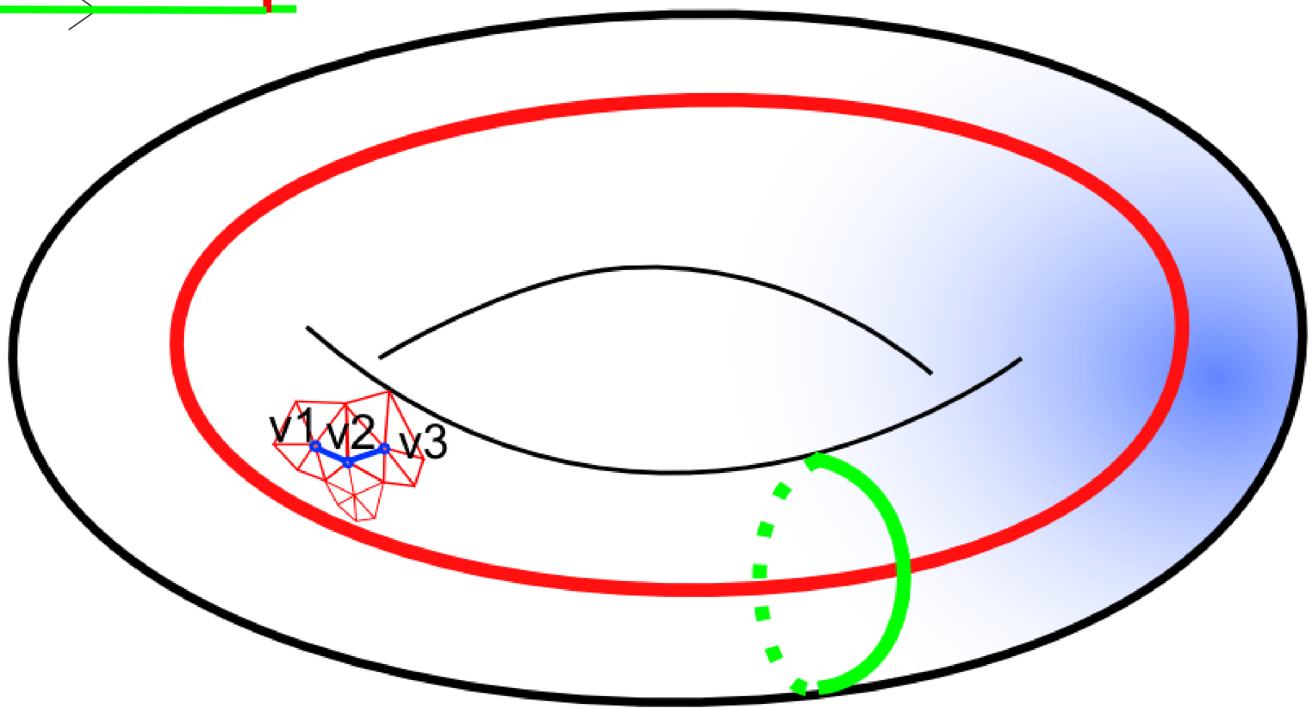

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7-We continue this process step by step.

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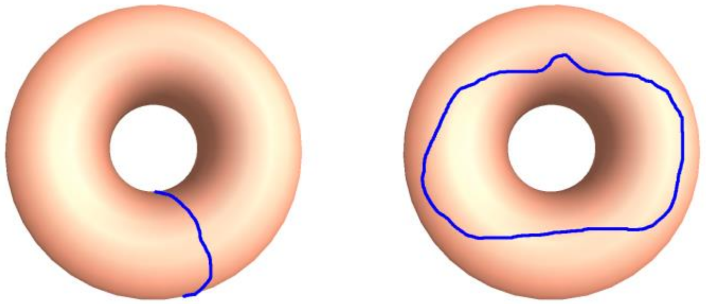
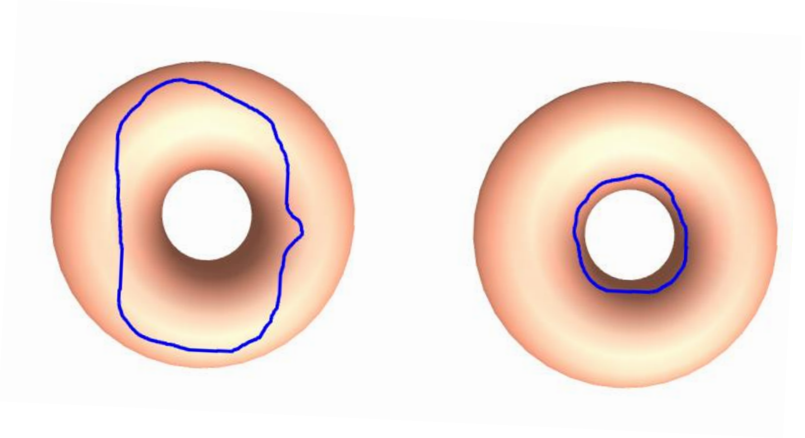
7-We continue this process step by step.

8-At the k-th step we can uniquely local the pre-image of v_k in $N(\hat{v}_k)$, $\hat{v}_k$, until we reach v_1 again.

Homotopy detection



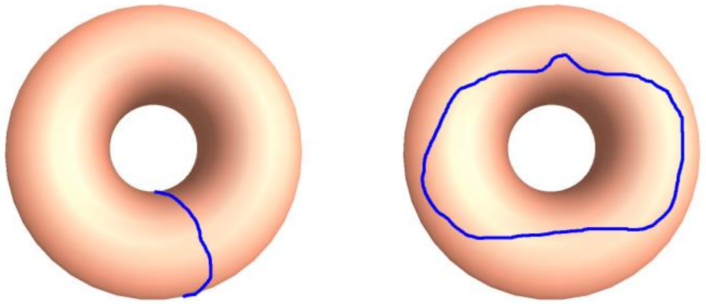
Non-homotopic curves



Homotopic curves

Image :David Gu

Homotopy detection



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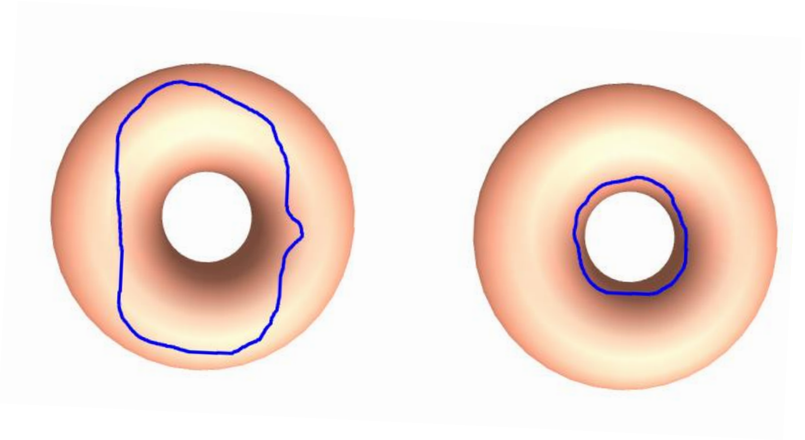

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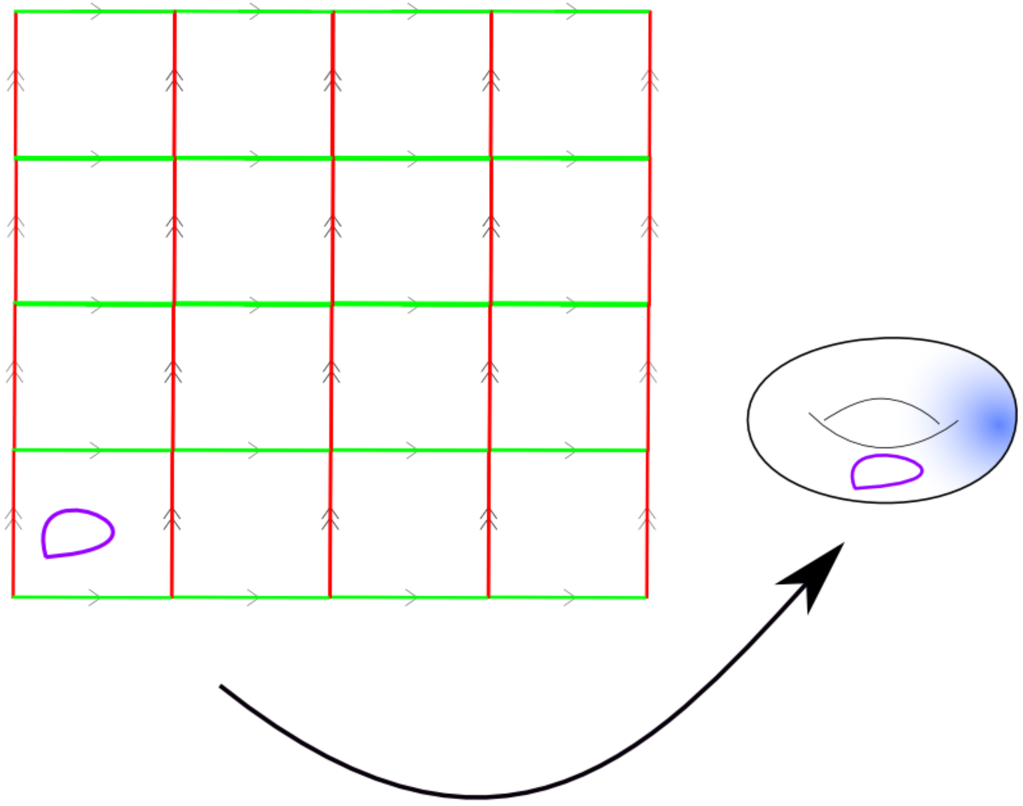

Homotopically trivial loops are lifted to closed loops in the covering space.

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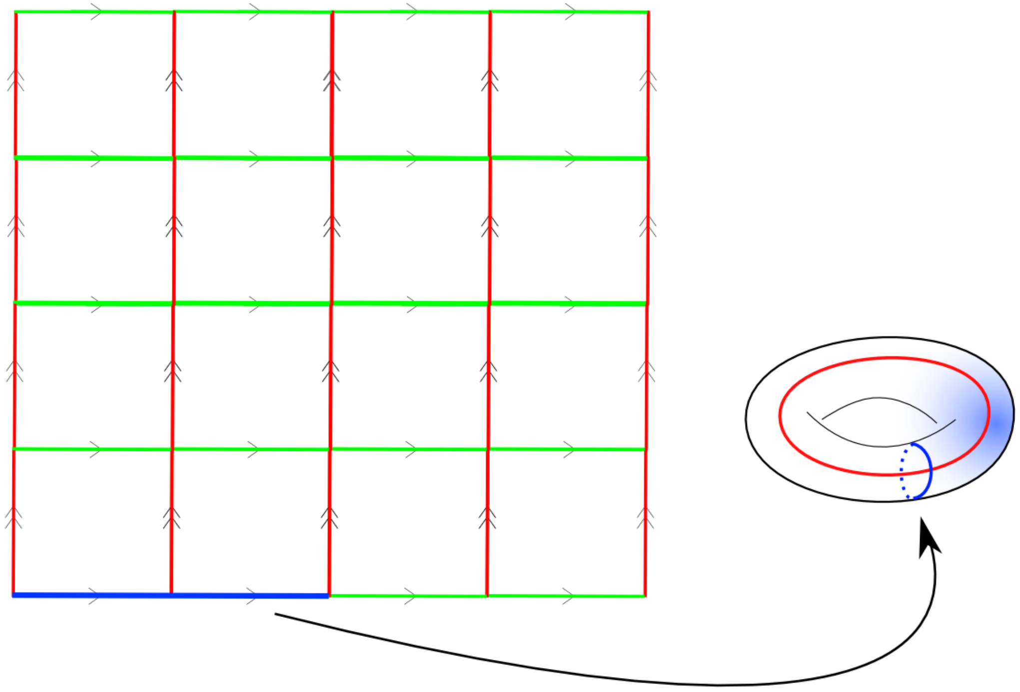

Homotopically non-trivial loops are lifted to open curves in the covering space.

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Output: Whether γ_1 is homotopic to γ_2

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5-Construct a finite portion of the universal cover.
6-Left Γ to the universal cover and obtain the curve $\bar{\Gamma}$. If $\bar{\Gamma}$ is a loop then γ_1 is homotopic to γ_2. Otherwise, they are not homotopic.

References

Algorithms presented here can be found in :

D. Gu and S Yau, Computational conformal geometry. Somerville, Mass, USA: International Press, 2008.